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POINT SETS IN THREE AND HIGHER DIMENSIONS AND THEIR INVESTIGATION BY MEANS OF A UNIFIED ANALYSIS SITUS*

BY R. L. WILDER

1. *Introduction.* When I was invited to give this symposium lecture, I hesitated somewhat to speak upon a field of mathematics which has already been represented before this Society by three Colloquia† and several lectures.‡ It occurred to me, however, that each of these lectures and Colloquia had been devoted to one of the two special "branches" of analysis situs, that is, either to combinatorial topology or to set-theoretic topology.§ (I think it is fair to state that most of the workers in analysis situs can be identified, by the evidence of their published works, with one of these "branches.") We have had no report before this Society which makes clear the relations between these schools of analysis situs—why there are two schools and what is the difference between them—nor is it evident that any but a few topologists are aware of the tendency, which has become manifest within the past few years, for the lines of demarcation between these "branches" of analysis situs to disappear. There are reasons, indeed, for believing that many topologists do not approve of this tendency, whether for esthetic reasons or because of their faith in the power of their own

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* Symposium Lecture delivered at the meeting of the Society at Chicago, April 8, 1932.

† O. Veblen, *Analysis Situs*, Colloquium Publications, vol. 5, Part II; R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications, vol. 13; S. Lefschetz, *Topology*, Colloquium Publications, vol. 12.

‡ R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 289–302; J. R. Kline, *Separation theorems and their relation to recent developments in analysis situs*, ibid., vol. 34 (1928), pp. 155–192; E. W. Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, ibid., vol. 33 (1927), pp. 13–34; T. H. Hildebrandt, *The Borel theorem and its generalizations*, ibid., vol. 32 (1926), pp. 423–474.

§ It might seem that in the case of Lefschetz' Colloquium, this remark is not true, since he considers topics that are ordinarily considered as part of the set-theoretic topology, for example, compact metric spaces. However, as we shall show, it is in method, not subject matter, that the two schools of topology differ, and in this sense Lefschetz' book is combinatorial.

methods, I do not know. However, such a tendency exists, particularly in the investigation of higher dimensional point sets, as well as in the study of compact metric spaces, and, in my opinion, has justified itself in the solution of problems which heretofore had defied the efforts of either the set-theoretic or the combinatorial methods, as well as in the providing of new and simpler solutions of older problems.

I hope I am justified, then, in what I propose to do, namely: (1) To maintain the thesis that there is only one branch of mathematics to be known as *analysis situs*, or *topology*, and that what are known as *set-theoretic topology* and *combinatorial topology* are really two *methods* within topology having the same aim. (2) To report upon the progress that has been made in the investigation of point sets in higher dimensions, and to demonstrate the tendency of the set-theoretic and combinatorial methods to unite in this investigation.

In carrying out these purposes, I shall divide my lecture into two parts. In the first part I shall give a brief introduction to the set-theoretic and combinatorial methods, trying to make clear what they are and what distinguishes them from one another, as well as what they can and what they apparently cannot accomplish separately.

In the second part, I propose to report on what has been accomplished in the topology of point sets in E_n $(n \geqq 3)$, with a view not only to furnish a summary of the present-day status of this study, but to show how the set-theoretic and combinatorial methods unite in this study to form what we might call a unified analysis situs.

Before concluding these introductory remarks, I wish to quote from two of the above-mentioned lectures. In concluding his lecture at Kansas on continuous curves from the viewpoint of analysis situs,* R. L. Moore stated: "Beyond certain propositions which hold true for both two and three dimensions, comparatively little is known concerning continuous curves in three-dimensional space. A fruitful field for future investigation is afforded in that branch of analysis situs which has to do with various special types of continuous curves such as simple continuous arcs, simple closed curves, and simple closed surfaces, in space of three dimensions. Little is known concerning the

* Loc. cit.

relation of such point sets to their complementary domains." The propositions which Moore referred to here as holding for both two and three dimensions, were, in the main, propositions concerning the internal properties of continuous curves which are now recognized to hold in general metric spaces, and have not to do with the relation of continuous curves, or, as I shall call them, Jordan continua, to their complementary domains. As to the relations of arcs, simple closed curves, and surfaces to their complementary domains, most of the fundamental problems concerning these types of Jordan continua have been cleared up since the date of Moore's lecture, as we shall see below. Furthermore, we are in possession of many details concerning the relations of general closed point sets to their complements in E_n, with which we were unfamiliar at the time of Moore's report, and we shall see that many of these relations, of necessity, take a combinatorial form such as is exhibited in the duality relations.

As recently as 1927, J. R. Kline stated in his concluding remarks of a lecture on separation theorems in analysis situs,* "As is evident from the matter presented in this paper, our knowledge of separation properties of sets immersed in three or more dimensions is comparatively limited. Here is an extremely interesting and important field that is practically untouched." We shall see that a great deal has since been accomplished in regard to these separation theorems in addition to the results which Kline mentions; as a matter of fact, we have a solution, by a unified analysis situs, of two of the main problems to which Kline refers in his lecture.†

In view of these remarks of Moore and Kline as to the general lack of knowledge concerning higher dimensional point sets, I think it may be particularly fitting to report upon the knowledge which we have recently gained—a knowledge which I believe is now considerable. We shall see, moreover, that many problems which not long ago seemed far beyond our scope, are now, thanks to a unified analysis situs, well within our grasp.

* Loc. cit.

† That is, the converse of the Jordan-Brouwer separation theorem in E_3 (see Kline, loc. cit., pp. 158–159), and the finding of necessary and sufficient conditions that the sum of two closed connected sets, neither of which separates E_3, may have a sum that separates E_3 (see Kline, loc. cit., p. 191, problem (b)).

I

2. *Definition of Analysis Situs.* In a paper which recognizes frankly the existence of two "schools" of analysis situs, we should first make clear just what analysis situs is. Despite the attention which this field has received from this Society, I am frequently assailed by the conviction that many of us do not know what analysis situs is, nor what it is trying to do. Some of the difficulty is due to the name itself, since the word "analysis" has, for most mathematicians, a very different connotation from what is intended here. We do not, as a rule, think of geometry when we hear the word "analysis." Yet analysis situs is geometry, differing from other branches of geometry only in the type of transformation which characterizes it, and in the great generality of the spaces which it often considers.

Perhaps this difficulty is a modern one, since the effect of Klein's *Erlanger Programm* seems to have clarified, in the mathematical mind of the past century, the distinction between that type of geometry which it called, variously, "topology," "geometria situs," "analysis situs," and other types of geometry. A recent speaker* before the Deutsche Mathematiker-Vereinigung suggests that Listing adopted the term "Topologie" in order to avoid a possible confusion with the "Geometrie der Lage" of von Staudt. Perhaps we should use the term "topology" today in preference to "analysis situs," in order to suggest things geometrical.

As a branch of geometry, analysis situs has distinguished itself by breaking away from the bounds that held other geometries to special types of spaces. The limit point or neighborhood notion led (Fréchet, Riesz, Hausdorff) to the concept of general topological or abstract spaces, a concept which may be said to form the foundation of modern topology. In terms of neighborhood, a point P is said to be a *limit point* of a point set M if every neighborhood of P contains at least one point of M distinct from P. Then two point sets A and B are said to be *homeomorphic* if there exists *a point for point correspondence between A and B which preserves limit points*; that is, a correspondence such that if M is a point set and P a point which lies in A,

* G. Feigl, *Geschichtliche Entwickelung der Topologie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 37 (1928), pp. 273–286.

and M' and P' are their correspondents, respectively, in B, then P is a limit point of M if and only if P' is a limit point of M'. This relation between A and B is also often expressed by saying that there exists a *topological transformation* of A into B (or B into A). Analysis situs, or topology, is *that branch of geometry whose objects of study are the properties of point sets that are invariant under topological transformations.*

Why do we have two "branches," or "schools," of analysis situs? If we were to glance at some of the literature on topology we might be led to conclude, judging from the differences in the problems studied, that there must be two aims or goals in analysis situs which lead to different types of problems. This is not the case. If one has a mathematical tool which will serve to attack a certain type of problem, he is very likely to seek that type of problem—no doubt this accounts for some of the literature on analysis situs. There is, in fact, only one goal in analysis situs, and this is indicated in the above definition, and this actual ultimate goal should not be crowded from our sight by what we seek as our immediate goals.

We shall find the answer to the question just proposed in the works of Cantor and his followers, and in the works of Riemann and Poincaré. Due to these, there have evolved two pathways to the ultimate goal, which we may call methods, and we shall see that it is of the set-theoretic *method*, instead of the set-theoretic branch or school that we should speak, and the same applies to the combinatorial method. First, however, let us view the fundamentals of these two methods.

3. *The Set-Theoretic Method.* The set-theoretic method is based directly on the neighborhood-limit point notion, and, in general, does not assume any such structural basis as the homeomorph of the fundamental cube of E_n. The intuitive notion of connectedness in geometric forms is given a precise formulation as follows. A set of points M is called *connected* if it is not the sum of mutually exclusive non-vacuous sets M_1 and M_2, neither of which contains a limit point of the other.* Regarding the properties of sets which are known to be connected, but about which we have no additional information, some things of im-

* This is the so-called Lennes-Hausdorff definition.

portance have been found,* but the theory at present in this regard is not very extensive, and to date does not yield to the combinatorial method at all. However, if we further restrict a set by requiring that it be *closed*, that is, that it contain all of its limit points, then we get what is known as the *continuum*, and a more extensive theory. Most of the early set-theoretic work in topology was done on continua, although there exists an extensive theory of sets which are closed, but not necessarily connected. Among the continua that have received considerable attention are the *Jordan continua*. Although characterized analytically by Jordan, they may be characterized by topological properties as follows. A set M is *locally connected* if, given any point P of M and a neighborhood U of P, there exists a neighborhood V of P contained in U such that all points of M in V lie in a connected subset of M which itself lies in U. A simple example of a set that is *not* locally connected is the set M in the plane consisting of (1) all points on the curve $y=\sin 1/x$ for $0<x\leqq 1$ and (2) all points (x, y) such that $x=0$ and $-1\leqq y\leqq 1$. Using as neighborhoods the interiors of circles, it is easily seen that M fails to be locally connected at all points where $x=0$. If, then, we require of a continuum that it be locally connected, we get a generalized Jordan continuum; the addition of the condition that it be *compact*, that is, that every infinite set in it have a limit point,† yields what is ordinarily known as the Jordan continuum. Most of the simple forms of continua, as the arc, the simple closed curve, the simple surfaces, etc., are Jordan continua; all of the euclidean n-dimensional spherical spaces are Jordan continua, and all of the euclidean spaces are generalized Jordan continua. The theory of Jordan continua is very extensive and has been developed almost exclusively by the set-theoretic method. The theory of Jordan continua that lie in the plane has been almost exhaustively developed, both with regard to the internal properties of the continua and as regards their external properties.

Because of their importance, these terms deserve further explanation. By the *internal properties* of a set, we shall mean

* See, for instance, B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255.

† In euclidean space this means simply that the continuum lies wholly within some sphere, that is, is bounded.

those properties which can be described by considering the set itself as the space; if it is a subset of another space, a neighborhood is defined as merely the set of all its points that lie in a neighborhood of the space in which it is imbedded. By *external properties* of a set we shall mean those properties which have to do with the relations of the set to the space in which it is imbedded; without this imbedding space no external properties exist. Thus, in the plane, if a Jordan continuum is the boundary of a domain, then all its points are *accessible* from this domain, that is, an optional point P of the continuum is joined to an optional point Q of the domain by an arc which has only P in common with the continuum. Another example: If a Jordan continuum M separates two points P and Q of the plane, then these points are separated by a simple closed curve of M. (As we shall see later, this is a special case of a general linking theorem.) Both of the theorems just stated are theorems concerning the external properties of a set. On the other hand, the theorem that if P and Q are points of a Jordan continuum M, then there exists an arc *in* M with the end points P and Q, is a statement concerning the internal properties of the Jordan continuum. Most of the theorems concerning internal properties of Jordan continua extend without any other change than that of terminology to higher dimensional euclidean spaces and to more general spaces. However, in the investigation of the *external* properties of point sets, particularly those that we shall call *properties in the large*, the set-theoretic method seems to have been lacking, and this is not surprising owing to its foundation in the neighborhood idea.

In closing this brief description of the set-theoretic method, I think that it will be highly appropriate if I quote from the introduction of Schoenflies' *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, an Ergänzungsband of the Jahresbericht der Deutschen Mathematiker-Vereinigung, published in 1908. A great deal of the set-theoretic work that has been done in this country was inspired by this work of Schoenflies. To quote: "No one will fail to recognize the immense usefulness which the arithmetic tendencies in the domain of geometry have achieved; the set-theoretic method is itself in a certain sense an offspring of arithmetization. Its ideas and methods have all the precision and rigor that characterize the arithmetic treat-

ment. But the demand to use in geometry exclusively analytical language and ideas would overshoot the goal. Moreover it entails the danger of one's forgetting to investigate more closely the spatial and formlike significance of the analytic symbols and therefore succeeding only to a partial knowledge.

"In fact, this is the development which we encounter on many occasions. We possess, to be sure, a very pure analytic, but a much less pure and precise geometric language · · · .

"In the following report one can easily designate two main groups of mathematical theorems · · · . A first group is formed by the general theorems on point sets; they represent the set-theoretic and accordingly the arithmetic foundation. A second group is formed by the simple theorems on straight lines, polygons, and polyhedrals, which I assume as given without a closer axiomatic analysis. In these is the conception of form, the intuitively accessible foundation, contained."

What are we to gather from these words of Schoenflies? In the first place he says clearly that in order to preserve a contact with geometry, and with intuitive conceptions of form, it is necessary to use, besides the set-theoretic method, theorems on polygons and polyhedrals. This he does, and since his work is almost exclusively laid in the plane, the properties of polygons are sufficient for his needs. However, at the end of Chapter 5 of this same work, a chapter in which he discusses topological invariants, and proves such theorems as the Jordan Curve Theorem, he says "I have chosen the methods of proof in this chapter in such a way that they permit an extension to configurations in space. Nevertheless the proofs are directly applicable only in a certain part. First, in three-space must be taken into consideration the contrast between curve and surface, and secondly, the connectivity number (referring here to the Riemann connectivity numbers) plays an important role in the theorems and proofs. The connectivity number is an invariant of topology, but without a knowledge of it only a part of the developments of this chapter can be directly extended. Yet one will not consider this a defect in the methods used. For the consideration of the connectivity number in three-space is unavoidable; every method which ignores it would yield only a part of that which is to be proved."*

* See also F. Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 335, last paragraph of §10.

Although many of Schoenflies' ideas concerning the nature of sets of points in three-space may look peculiar in the light of our more advanced knowledge, I believe that we may well give thought to the words which I have quoted. Instead of saying today, as Schoenflies said twenty-four years ago, "We possess a very pure analytic, but a much less pure and precise geometric language," we might well say, "We possess a pure set-theoretic language and a precise and rigorous theory of the connectivity of higher dimensional polyhedrals. In order to extend the theorems of the plane to higher dimensions, it is necessary that we combine these two in our investigations." I am confident that this follows the spirit of Schoenflies' ideas. In general, however, the set-theoretic investigators have followed the *path* indicated by the researches of Schoenflies in the theory of sets of points, but have not retained his general *method*, nor heeded his suggestions relative to extending his results to three-space.

I have given some idea of the set-theoretic method, which proceeds from the notion of limit point and neighborhood. The theory of the connectivity numbers of higher dimensional polyhedrals, or complexes as we shall call them, has been developed by the combinatorial method, to which I shall now turn.

4. *The Combinatorial Method.* There are, broadly speaking, two kinds of combinatorial topology, the *pure* and the *geometric*. Although, in dealing with internal properties of sets of points, we shall use chiefly the pure combinatorial topology, it is probably better that we consider the combinatorial method from the standpoint of the geometric type, since it is the more intuitive and less abstract and of great use in the investigation of external properties; formally, however, the difference is superficial.

In the geometric type of combinatorial topology the element with which we operate is the *n-cell*, which for our purposes* we consider as a point set which is homeomorphic with the n-dimensional tetrahedron. The *boundary cells* of the n-cell are those i-cells $(0 \leqq i \leqq n-1)$ that are the homeomorphs of the i-faces of the tetrahedron. For the particular cases $n=0$ and $n=1$, we have, respectively, the point (no boundary cells) and the arc (boundary cells are two 0-cells). For $n=2$, we have

* As a rule, the n-cell is defined to be the homeomorph of the *interior* of the tetrahedron.

the homeomorph of the plane triangle together with its interior (boundary cells are three 1-cells and three 0-cells). An *n-complex* is a configuration consisting of a collection of n-cells, called *fundamental cells* of the complex, where two cells of the collection either have no point in common, or have only a boundary cell of each in common. According to how we define the *boundary* of a complex we get various types of theories.* Formally speaking, if K^n is an n-complex, we may represent each $(n-1)$-cell, which is a boundary cell of at least one fundamental cell of K^n, by a symbol, and denote the boundary of each fundamental cell of K^n by a certain algebraic expression in these symbols; then the boundary of K^n is, say, the association of the algebraic sum of these expressions with the cells whose symbols occur in it. In the modulo 2 analysis situs, the boundary of the n-cell is denoted by the linear expression of the symbols for the boundary $(n-1)$-cells with coefficients unity; the algebraic sums are taken modulo 2, and thus the boundary of K^n is the set of those $(n-1)$-cells that are boundary cells of an *odd* number of the fundamental cells. If the sum, modulo 2, of the boundaries of the fundamental cells of K^n is zero, that is, if each $(n-1)$-cell is on the boundary of an even number of the fundamental cells, then K^n is an *n-cycle*, modulo 2.† If no proper subset of the fundamental cells of an n-cycle K^n forms an n-cycle, and also every point of K^n is in a neighborhood (in the set-theoretic sense) of the complex which with its boundary (set-theoretic) is an n-cell, then K^n is an *n-manifold*, modulo 2.

Although, in the classical combinatorial topology, orientation of cells is introduced, and Alexander has introduced a modulo m topology, we may limit ourselves for purposes of introduction to the modulo 2 type; particularly since we intend no complete survey of combinatorial topology, but rather propose to indicate only the general method.

The simple closed curve when subdivided into 1-cells so as to form a complex, is a 1-manifold, the only one, in fact. Among the 2-manifolds are the 2-sphere, or the simple closed surface as the set-theorists call it; also the torus, and, in general, the simple closed surface with any number of handles attached. We notice,

* See Lefschetz, *Topology*, Chap. I, §14.

† In particular, the boundary of an n-cell is an $(n-1)$-cycle, indeed, an $(n-1)$-sphere. A 0-cycle is any even number of 0-cells.

on the torus, that there can be drawn two types of simple closed curves neither of which is the boundary of any collection of the 2-cells (however these be represented) that make up the torus. This is a topologically invariant property of the torus, and we express it by saying that the *Betti number* of the torus of dimension 1, or simply the 1-dim. Betti number, is 2. The 1-dim. Betti number of the sphere is zero. This brings us to the notion of a *homology*. On a complex K two i-cycles are said to be *homologous* to one another if together they bound some $(i+1)$-complex of K. If we denote the i-cycles by C_1^i and C_2^i, we express symbolically the fact just stated as follows:

$$C_1^i \sim C_2^i \qquad \text{(modulo 2, } K\text{)}. \tag{1}$$

If an i-cycle C^i of K itself bounds some $(i+1)$-complex of K, we write

$$C^i \sim 0 \qquad \text{(modulo 2, } K\text{)}, \tag{2}$$

to be read "C^i is homologous to zero on K, modulo 2." On the 2-sphere, if $i=1$, (2) holds for every C^i. As we have seen above, this is not true on the torus. Let us pick out two simple closed curves on the torus of the types mentioned above as not bounding on the torus. Denote them by C_1^1 and C_2^1. Then if C^1 be another 1-cycle on the torus, we shall find that there exists a sum $C^1+\eta_1 C_1^1+\eta_2 C_2^1$,* where η_1 and η_2 are either 1 or 0, which bounds some 2-complex on the torus. As we can express this fact by a relation of type 2, we say that the three 1-cycles are not *linearly independent with respect to homologies* on the torus.

We are now ready to generalize further the notions just introduced. On a complex K let $C_1^i, C_2^i, \cdots, C_k^i$ be distinct i-cycles having the following properties: (1) there exists no relation of the form $\eta_1 C_1^i+\eta_2 C_2^i+\cdots+\eta_k C_k^i \sim 0$ (modulo 2, K) unless the η's are all zero, that is, these i-cycles are linearly independent with respect to homologies on K; (2) if C^i is any i-cycle of K, then either it is one of the cycles of the above set, or it satisfies a relation of the form $C^i \sim \eta_1 C_1^i+\eta_2 C_2^i+\cdots+\eta_k C_k^i$

* All sums are modulo 2; thus, the sum of a number of n-complexes is the collection of those n-cells (and their boundaries) that are in an odd number of the complexes. It is easy to see that the sum of a finite number of n-cycles is itself an n-cycle (or vacuous); also, the boundary of an n-complex is an $(n-1)$-cycle (or vacuous).

(modulo 2, K), where the η's are either 1 or 0. Then the above set of i-cycles is said to form a *basis* for the i-cycles of K or i-basis of K, and their number, k, is called the i-dim. (or ith) Betti number of K. Symbolically expressed, this fact is written $p^i(K)=k$.

We can also define Betti numbers for the complement of a point set in E_n. For purposes of simplification, as well as of symmetry, let us consider the n-dimensional spherical space H_n, that is, the set of points $x_1^2+x_2^2+\cdots+x_{n+1}^2=1$ in E_{n+1}. By an easily defined procedure of *cutting* up (by passing n-planes through H_n*), H_n may be subdivided into a finite number of n-cells as small as we please, whose sum makes up H_n and in terms of which H_n is an n-complex. Let us adopt once and for all a set S of subdivisions of H_n, namely, $S_1, S_2, S_3, \cdots$, such that for each i, S_{i+1} is a subdivision of S_i and such that if e is any positive number, there is an i for which the cells of S_i are all of diameter less than e. Let M be any closed set of points in H_n. If C^i is a j-cycle of S_i which lies wholly in H_n-M, then C^i is said to be homologous to any cycle of $S_{i+h}(h>0)$ which is a mere subdivision of C^i; and if in any S_{i+h} there exists a complex K^{i+1} whose cells lie wholly in H_n-M and whose boundary is C^i (or C^i subdivided into finer cells), then C^i is said to be homologous to zero in H_n-M; that is, $C^i\sim 0$, (H_n-M). If no such complex K^{i+1} exists in any S_{i+h}, then C^i is said to be unbounding in H_n-M, or to *link* M in H_n. The definitions of homologies between cycles in H_n-M, and of linear independence (in H_n-M) between cycles with respect to homologies, are given as before, keeping in mind that the complexes considered must at all times lie in H_n-M. If we now denote by $p_i^j\ (H_n-M)$ the maximum number of independent (in H_n-M) j-cycles of S_i, then the number $p^j(H_n-M)=\mathrm{Lim}_{i\to\infty}p_i^j(H_n-M)$, which may be finite or infinite, is called the jth Betti number of H_n-M. *This number is invariant for all possible choices of the sequence S, as well as for all topological transformations of M in H_n.* In particular, the number $p^0(H_n-M)+1$ represents the number of domains complementary to M, and as a consequence of the theorem just stated, the number of domains complementary to a closed set in H_n (and in E_n for M bounded) is an

* See J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Transactions of this Society, vol. 23 (1922), pp. 333–349.

invariant external property of M. The proof of this theorem is one of the achievements of unified analysis situs.

The case where M, instead of being any closed set, is a complex, in the sense previously defined (that is, not necessarily formed from cells of the S_i's), is not only interesting but important for what follows. Connecting the Betti numbers of M and of $H_n - M$, we have the fundamental Alexander Duality Theorem:*

$$p^i(M) = p^{n-i-1}(H_n - M), \qquad (0 \leqq i \leqq n-1). \tag{3}$$

Thus, where the Betti numbers of M are known, we know the Betti numbers of $H_n - M$, and conversely. A special case of this theorem is that (chief) part of the classical Jordan-Brouwer separation theorem, proved by Jordan and others for the plane (Jordan Curve Theorem) and by Brouwer for the general case, which states that a topological $(n-1)$-sphere in H_n separates H_n into just two connected domains; to get the latter theorem we let $i = n-1$ in (3), the Betti number $p^{n-1}(M)$ being equal to 1 since M is the only (unbounding) $(n-1)$-cycle in itself. For the cases where M is a topological i-sphere in H_n, $p^i(M) = 1$, and $p^s(M) = 0$ where $s \neq i$, and there is therefore just one cycle in $H_n - M$ that does not bound in $H_n - M$, this cycle being of dimensionality $n-i-1$; thus the i-sphere is linked by just one independent cycle.

It has been shown independently by Pontrjagin and Frankl that (where M is a complex) the i-basis of M and the $(n-i-1)$-basis of $H_n - M$ may be so chosen that the cycles of the two bases link uniquely; that is, so that a cycle of the i-basis of M links one and only one cycle of the $(n-i-1)$-basis of $H_n - M$, and conversely.

In the pure combinatorial method, instead of dealing with the n-cell as the homeomorph of the n-dimensional tetrahedron, we consider any set of $n+1$ points as forming the n-cell. The ${}_nC_{i+1}$ sets of $i+1$ points of such a set form the i-cells that enter into the boundary of the n-cell. The definitions of complex, cycle, etc., are given as before. The pure combinatorial method is of particular value in studying the topological properties of general closed point sets. Because of the foundation upon finite sets

* See J. W. Alexander, loc. cit.

of points its application is not restricted to subsets of euclidean spaces.

5. *Contrast of the Two Methods.* I have tried to indicate the general nature of the set-theoretic and combinatorial methods, the one based on the neighborhood-limit point notion, the other based essentially on the euclidean tetrahedron. If every topologist used both of these methods, there would be hardly any *raison d'être* for the present report. As I have already stated, however, topologists in general are either *set-theoretic* or *combinatorial*; the consequences of this situation may as well be frankly stated: a too prevalent tendency on the part of one to ignore the works of the other; the inability to extend many results of the set-theoretic topology to higher dimensions; overlapping of results; and inexcusable delay in the discovery of important general theorems.

Why is it necessary, as Schoenflies seems to have recognized, to use both methods in the investigation of point sets in higher dimensions? The answer, I believe, lies in the difference between what we may call *local properties*, and *im grossen properties* or *properties in the large*. We may say that a local property, as distinguished from a property in the large, is a property that concerns the arbitrarily small neighborhoods of a point.

Glance at the literature on set-theoretic topology and notice the preponderance of local properties. The property of local connectedness is basic in the theory of Jordan continua. The Menger-Urysohn theory of dimension is based on a local property. On the other hand the combinatorial method deals essentially with properties in the large. As an example, the theory of linking, which occupies a central position in combinatorial topology, is based upon a property in the large.

To be sure, either of the above methods can apparently be extended so as to take care of both types of properties in many cases. The combinatorial method, by proceeding from the finite by a limiting process, somewhat as we proceed from the notion of a sum of a finite series to the sum of an infinite series, can be so expanded that it can take care of all the topology of compact metric spaces.* Whether it is the easier method in a given

* See P. Alexandroff, *Simpliziale Approximationen in der allgemeinen Topologie*, Mathematische Annalen, vol. 96 (1926–27), pp. 489–511. Also see *Gestalt und Lage*, referred to below.

problem is beside the point here; we can completely characterize, topologically, a compact metric space by combinatorial methods. This procedure, the spirit of which is to be found in Schoenflies' work, but whose modern form finds its inception in the work of Brouwer* and in general form in the work of Vietoris,† Alexandroff,‡ and others, will be explained later.

On the other hand, the very notion of *connected* itself, which as we have seen is basic in the set-theoretic topology, is a property in the large. So also is the property of *nth degree connectedness*, introduced by Menger.§ Furthermore, in the proceedings of the 1929 congress of Slavic mathematicians, Knaster¶ has introduced certain properties based on the notion of connectedness, which are clearly properties in the large, and which Knaster himself suggests may lead to a theory which can accomplish all that the combinatorial method has accomplished.

Despite these facts, I think it fair to say that to date the chief power of the set-theoretic method has been manifest in the investigation of local properties, whereas that of the combinatorial method has been apparent in the investigation of properties in the large. Thus, in the topology of the euclidean plane the set-theoretic method seems to have been able to attack almost any kind of problem, since here the properties in the large were restricted by the very nature of the space; in the plane a closed curve cannot be knotted, nor can two closed curves link one another, for instance. In higher dimensions, however, due to the greater degree of freedom of the space, the *im grossen* properties of a set become so complicated that the set-theoretic method has not been successful, by itself, in treating the prob-

* L. E. J. Brouwer, *Beweis der Invarianz der geschlossenen Kurve*, Mathematische Annalen, vol. 72 (1912), pp. 422–425.

† L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Mathematische Annalen, vol. 97 (1927), pp. 454–472.

‡ See P. Alexandroff, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Mathematics, vol. 30 (1928–29), pp. 101–187, as well as earlier papers referred to therein. The present citation will hereafter be referred to as *Gestalt und Lage*.

§ See K. Menger, *Dimensionstheorie*,1928, p. 214, §4.

¶ B. Knaster, *Einige Probleme über Punktmengen mit Fixpunkten*, Comptes-Rendus du I Congrès des Mathématiciens des Pays Slaves, Warszawa, 1929 (Warsaw, 1930), pp. 287–290. See also K. Borsuk, *Quelques théorèmes sur les ensembles unicohérents*, Fundamenta Mathematicae, vol. 17 (1931), pp. 171–209.

lems which arise; and until the set-theoretic method has so expanded itself as to take care of these properties, it seems logical to call upon the combinatorial method for assistance.

Similarly, the combinatorial method, to date, is restricted in its application to closed sets of points, or to compact spaces, and there is no evidence, as yet, that it can be expanded so as to take care of the topology of non-compact sets, excepting, of course, certain particular cases such as the complements of closed sets in E_n (a special case of *infinite complexes*).*

Is it possible for either method to become supreme in analysis situs? This question, which I think is suggested by the remarks in the preceding paragraphs, can be answered only by prophecy, and I shall not presume to prophesy. I think it fair to say, however, that if topologists will continue to restrict themselves to one method, the case for the set-theoretic method looks the more hopeful, although the difficulties encountered may be too great to make the use of the method expedient. Perhaps the set-theoretic method is as yet only in its infancy; its application chiefly to the study of local properties, due to its foundation in the neighborhood idea, may be only a sign of youth. That this is not necessarily final, however, is indicated by the fact that every separable metric space can be imbedded in a compact metric space of the same dimension, and as already stated a compact metric space is amenable to combinatorial treatment.

However, until the set-theoretic method has offered a substitute, say, for the linking theorems of the combinatorial topology, and until the combinatorial method has offered a suitable method for dealing with properties of non-compact sets, I believe we must have a unified analysis situs. It is where the two methods seem to meet common ground, where one appeals to the other for assistance, that we have to deal with a unified analysis situs.

6. *Evolution of the Two Methods.* As is the case with many other branches of mathematics, topology had its beginnings as a means to certain ends, and it is in the beginnings of topology that we have to look for the introduction of the set-theoretic and combinatorial methods. As I have already intimated, it is in the works of Riemann and Poincaré that we find the inception of

* See Lefschetz, *Topology*.

the combinatorial method. Riemann found that for the investigation of functions which arise from the integration of total differentials, it was necessary to differentiate between the various connectivities of simple surfaces; in Weber's edition of Riemann's *Werke* will be found a paper dealing with the notions of simply connected and n-fold connected surfaces, and in a *Fragment* will be found the essential notions embodied in the numbers of Betti (Betti's work in this connection was done without the knowledge of this *Fragment*, which later appeared in Weber's edition of Riemann's works). Poincaré had already employed topology in the study of differential equations when he wrote his classical memoirs on analysis situs. In these memoirs he evidently had in view principally the classification of algebraic surfaces; indeed the last two deal with the applications of analysis situs to algebraic geometry, an application that is still attracting much interest.* The combinatorial method, with its theory of connectivity and applicability in general to properties in the large, is the natural outcome of the type of problem to which Riemann and Poincaré adapted it.

As for the set-theoretic method: The neighborhood and limit point notion at the basis of the method of course formed the foundation of the point set theory as developed by Cantor and his followers, whose work looked mainly to the foundations of modern analysis. Cantor developed many theorems, which are fundamental in the present-day set-theoretic topology. An important landmark in the evolution of the set-theoretic topology, one which seems to have given it a definite direction, is the work of Schoenflies from which I have already quoted. The influence of this work on the set-theoretic work in this country and Poland is very marked, as for instance in the case of Jordan continua, the first topological investigation of which is to be found in the work of Schoenflies. It seems to have been Schoenflies' object, as indicated in the introduction of his book, to show that purely geometric problems can be treated by purely geometric methods, and he attacked the Jordan continuum, for instance, as a geometric rather than as an analytic problem. That he was correct in his viewpoint is amply attested to by the immense literature that has accumulated on the theory of Jordan con-

* See S. Lefschetz, *L'Analysis Situs et la Géométrie Algébrique*, Paris, 1924.

tinua, giving such detailed analysis of their structural properties as would hardly have been possible by the analytic method.

Although Schoenflies' investigations were devoted to the plane (so far as his principal topological results are concerned), it is noteworthy that he did not, as do the modern set-theorists to whom he gave stimulus, employ a purely set-theoretic method. Modern set-theorists have shown rather conclusively that the set-theoretic method is sufficient for most of the topology of the plane. However, Schoenflies made considerable use of the properties of polygons and of the Riemann connectivity numbers of domains bounded by polygonal configurations. Thus, the properties of domain boundaries were investigated by considering them as limiting properties of the properties of approximating polygons. Furthermore, he indicated, as quoted above, that to proceed with the study of point sets in three dimensions, it would be necessary to have a theory of the connectivity of higher dimensional polyhedra. Accordingly I feel that it is in Schoenflies' work that we find the beginnings of a unified analysis situs.

However, it is apparent that the actual motivating factor which influenced the adoption of a unified method in topology was the work of Brouwer. Brouwer was clearly influenced in his selection of problems by the work of Schoenflies, whose work he both corrected and extended. In a paper published in 1912,* he showed, using a method which has recently been generalized to higher dimensions by Vietoris, that the number of domains complementary to a continuum in the plane is a topological invariant, and that the property of being a closed curve, as defined by Schoenflies, is a topological invariant. This paper, written to supply fundamental omissions in Schoenflies' work, contains the germ of the modern adaptation of combinatorial methods to general closed sets of points, which until recently were considered open to attack only by the set-theoretic method.

In general, however, we have had two "schools" in analysis situs, the allocation of a topologist to a "school" being determined by the *method* that he employed. Using the set-theoretic method alone the topologist has been able to clear up the topology of the plane both as to internal and external properties, and to prove a host of internal properties of point sets in general

* Loc. cit.

spaces—although even here, when properties in the large were concerned, the set-theoretic method has not yielded results. Using the combinatorial method alone, the topologist has been able to set up a series of invariants, especially numerical invariants such as the Betti numbers, for n-dimensional complexes, but has succeeded in classifying manifolds only through the two-dimensional case. The situation in combinatorial topology is indicated by two reports before the Deutsche Mathematiker-Vereinigung to which I refer below.* For the situation in set-theoretic topology I can refer the reader to the articles by Moore and Kline to which I have referred above.†

Regarding the theory of sets of points, to which I propose to devote the remainder of this report, the set-theorist has not, in general, been able to establish external properties and properties in the large in higher dimensions. It is to this unified analysis situs that we now turn. As we proceed, it may occur to the discerning reader how ironic it is, that in the works of Poincaré and Brouwer we are today finding the tools for our attack upon that branch of mathematics, the theory of sets of points, about which Poincaré stated "later generations will regard the theory of sets of points as a disease from which one has recovered," and with which, in the sense in which we conceive of it as a formalistic theory, Brouwer will to-day have nothing to do!

II

7. *General Methods.* Two general kinds of unified analysis situs seem to have been employed to advantage. Suppose S is a space and M is a point set in S. In the case where S is a euclidean space, I have found that to use the set-theoretic method *in* M, and the combinatorial method *in* $S-M$, has been quite successful in attacking such problems as the converse of the Jordan Curve Theorem in E_2, and investigating external properties of Jordan continua in E_n. However, the method which Vietoris has introduced employs the combinatorial method in M by setting up certain cycles as in the pure combinatorial

* G. Feigl, loc. cit., and B. L. van der Waerden, *Kombinatorische Topologie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 39 (1930), pp. 121–139.

† See also H. Tietze and L. Vietoris, *Beziehungen zwischen den verschiedenen Zweigen der Topologie*, Encyklopädie der Mathematischen Wissenschaften, vol. III_1, Heft 10 (1931).

topology together with certain relations between these cycles, and, for instance, their topological limits in M.

It has occurred to me that I might arrange this report according to results obtained by these two methods. However, as I wish to give a general survey of the situation regarding point set theory in higher dimensions, it will perhaps be more satisfactory to proceed according to subject matter.

8. *Homeomorphism Problems; Topological Definitions of E_n.* Under *homeomorphism problems* we group those problems that concern the finding of topological properties that are necessary and sufficient in order that there exist a homeomorphism between two sets of points. For instance, if each of two sets of points is (1) a compact space and (2) irreducibly connected between two of its points, then they are homeomorphic; indeed each is homeomorphic with the set of points on the linear interval $0 \leqq x \leqq 1$, and is therefore what we call an *arc*. The sum of two arcs with end points in common, and only those points in common, is an H_1. Thus, of basic importance among the homeomorphism problems is the characterization of the euclidean spaces, or, preferably, of the spaces H_n (see §4). Any set of points which is homeomorphic with an H_n is, in the analysis situs sense, itself an H_n.

Of course the topological properties which we have to impose upon a set (consisting of at least two points) in order that it be an H_n are partially dependent upon whether we are considering the set as a general topological space in itself, or as a subset of another space whose properties we are given. Suppose, for instance, that we are dealing with subsets of locally compact spaces; we can state the two following sets of properties as characterizing the H_n indicated:

H_1: Connected, locally connected, not disconnected by the omission of any $\left\{\begin{matrix}\text{point}\\ \text{0-cell}\end{matrix}\right\}$, but disconnected by the omission of any $\left\{\begin{matrix}\text{two points}\\ \text{0-sphere}\end{matrix}\right\}$.*

H_2: Compact, connected and locally connected, not disconnected by the omission of any $\left\{\begin{matrix}\text{arc}\\ \text{1-cell}\end{matrix}\right\}$, but disconnected by the

* See R. L. Wilder, *Concerning simple continuous curves and related point sets*, American Journal of Mathematics, vol. 53 (1931), pp. 39–55.

omission of any $\left\{\begin{matrix}\text{simple closed curve}\\ \text{1-sphere}\end{matrix}\right\}$.*

Both of these characterizations have been obtained by the set-theoretic method, and are thus ordinarily stated by reading the upper line in the braces. I have, however, indicated in the lower lines of the braces the corresponding combinatorial terms; perhaps we should, instead of "0-sphere" and "1-sphere," say "0-manifold" and "1-manifold." The following problems suggest themselves: Writing "H_0" for "0-sphere," etc., do we get an inductive definition of H_n? Assuming the definition of H_i, $i<n$, the $(n-1)$-cell is one of the parts into which an H_{n-2} divides H_{n-1}; thus, is a Jordan continuum which is not disconnected by the omission of any $(n-1)$-cell, but is disconnected by the omission of any H_{n-1}, an H_n? Obviously not in the general case, at least not without the existence in the set of an H_{n-1}. In particular, then, what conditions in addition to being a Jordan continuum which contains an H_2, not disconnected by the omission of any 2-cell but disconnected by the omission of any H_2, are necessary and sufficient in order that we have an H_3?

The most elementary definition of H_n was indicated by Tietze,† and is based upon the consideration of H_n as the product space of the already topologically defined H_1. A distinct disadvantage, perhaps, of this definition, is that the neighborhoods must be defined in terms of sets of n neighborhoods of H_1.

The definition of H_n given by Alexandroff‡ starts with a general topological space S, and depends upon the existence, in S, of a sequence (*spectrum*) of finite sets of points, F, F_1, F_2, $\cdots$, F_k, $\cdots$, called *nets*. The initial set F consists of $n+2$ points, and every subset of F consisting of $n+1$ points is called a *scaffold*; thus the points of F correspond to the $n+2$ vertices

* See L. Zippin, *On continuous curves and the Jordan curve theorem*, American Journal of Mathematics, vol. 52 (1930), pp. 331–350.

† H. Tietze, *Über Analysis Situs*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 2 (1923), pp. 37–68. If X and Y are spaces, the product space $X \cdot Y$ is a space whose points are the pairs (x, y) of points of X and Y, and for which a neighborhood of a point (x_0, y_0) is the set of points (x, y) such that x and y belong respectively to neighborhoods of x_0 and y_0. See also K. Yoneyama, *The conception of a curve, a surface and a solid*, Memoirs of the Kyoto College of Science and Engineering, vol. 5 (1912–13), pp. 261–269, for an inductive building up of the euclidean element.

‡ P. Alexandroff, *Zur Begründung der n-dimensionalen mengen-theoretischen Topologie*, Mathematische Annalen, vol. 94 (1925), pp. 296–308.

that we get in the division of H_n into $n+2$ n-cells. The set F_1 consists of $(n+2)\cdot(n+1)!$ scaffolds such as we get upon the barycentric subdivision* of the above cellular subdivision of H_n. In general, each F_k consists of scaffolds of $n+1$ points each, and the spectrum is so related to the space S by conditions on the closures of any scaffold of F_k and its *derived* (analogous to barycentric subdivision) scaffolds, that the space S is homeomorphic with the n-dimensional space H_n. It will be noticed that although this definition of H_n is entirely set-theoretic in form, its spirit is combinatorial in the sense that it preserves the analogy to the cellular subdivision of H_n and its continued barycentric subdivision. In short, it translates a fundamental combinatorial method into set-theoretic form, and is in this sense an achievement of a unified analysis situs.

Unfortunately limitations of space do not permit my going into detail concerning various definitions of H_2 other than that of Zippin given above.† And as to the progress that has been made on the homeomorphism problem relative to the n-dimensional manifold, I may refer my readers to the report by van der Waerden already cited above.

9. *External Properties of the Arc, Simple Closed Curve and Simple Closed Surface in* $E_n(n>1)$. I have indicated above, in quoting from Moore's Kansas lecture, that the fundamental problems here have been largely cleared up. In the case of the simple closed surface (that is, H_2), Brouwer‡ showed in 1912 that an $(n-1)$-dimensional manifold imbedded in E_n determines two domains, of which it is the common boundary, and in an accompanying paper§ established the accessibility of the manifold. Furthermore, since the sets mentioned in the heading

* Or *regular* subdivision; see Veblen, loc. cit.

† See R. L. Moore and J. R. Kline, this Bulletin, vol. 28 (1922), p. 380, abstract No. 7; T. Radó, *Über den Begriff der Riemannsche Fläche*, Acta Litterarum ac Scientiarum, Szeged, vol. 2 (1925), pp. 101–121; I. Gawehn, *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98 (1928), pp. 321–354; C. Kuratowski, *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), pp. 307–318; J. H. Roberts, *A point set characterization of closed 2-dimensional manifolds*, ibid., vol. 18 (1932), pp. 39–46.

‡ L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematischen Annalen, vol. 71 (1912), pp. 314–320.

§ L. E. J. Brouwer, *Über Jordansche Mannigfaltigkeiten*, ibid., pp. 320–327.

of this section are, respectively, the 1-cell, H_1, and H_2, and accordingly complexes in the combinatorial sense, the duality theorem of Alexander explained above (§4) embodies fundamental results concerning their external properties. As the Betti numbers of the 1-cell are all zero, the arc neither separates E_n nor is it linked by any cycle in its complement. The Betti numbers of H_1 being all zero, except p^1, which is 1, the simple closed curve is linked only by one cycle, an $(n-i-1)$-cycle, of its complement, that is, it separates E_2 into just two domains (see the Jordan Curve Theorem); in E_3 it does not separate space but is linked by a 1-cycle, etc. In the case of H_2 (simple closed surface), the Betti numbers are all zero except p^2 which is equal to 1, and the reader can easily supply his own conclusions as to space separation and linking.

The accessibility of the arc, as well as the accessibility of the i-cell $(i<n)$ imbedded in E_n, is a special case of a theorem of Mazurkiewicz* to the effect that any closed subset of E_n which is homeomorphic with a subset of E_{n-1} is accessible from its complement. Indeed, I have been able to show,† by strong use of the combinatorial properties of the complement of the set, that such a set is *regularly* accessible from its complement.

The theorem that the domains determined by a simple closed curve in E_2 are uniformly locally connected, established by Moore, I have extended to E_n by determining that the domains bounded by an $(n-1)$-manifold are uniformly locally connected.‡ This theorem is also capable of extension in another direction (see §10).

It will be admitted, I think, that our knowledge of the external properties of the arc, simple closed curve, and surface in E_n is today quite extensive. Thus, in E_3, the simple closed surface is the common boundary of just two domains, from each of which it is accessible, and each of which is uniformly locally connected, and is not linked by any 1-cycle of its complement (for the converse of this theorem see §10), properties which have been found by the combined use of set-theoretic and combina-

* S. Mazurkiewicz, *Sur un problème de M. Knaster*, Fundamenta Mathematicae, vol. 13 (1929), pp. 146–150.

† R. L. Wilder, *Extension of a theorem of Mazurkiewicz*, this Bulletin, vol. 37 (1931), pp. 287–293.

‡ R. L. Wilder, *A converse of the Jordan-Brouwer separation theorem in three dimensions*, Transactions of this Society, vol. 32 (1930), pp. 632–657, §1.

torial methods. To be sure, the theorem of Schoenflies regarding the extension of the homeomorphism between two simple closed curves of E_2 to the entire plane does not go over to H_2's in E_3,* but certainly we are not warranted in saying that the point set theorems of the plane do not extend to higher dimensions.

10. *The Jordan Continuum; Extension of Schoenflies' Results to Higher Dimensions.* For some time it has been an open question with the set-theorist as to just how much may be done with the theory of Jordan continua in E_n $(n>2)$, particularly in the matter of external properties. If the set-theoretic method alone is used, the prospects do not seem hopeful, but from the standpoint of unified analysis situs I believe a great deal can be done in extending the theory that Schoenflies built up in the plane.

One of Schoenflies' best known results is the converse of the Jordan Curve Theorem. Recognizing that not all closed curves are simple closed curves, Schoenflies set himself the problem of so extending the Jordan Curve Theorem as to get conditions that are sufficient, as well as necessary, in order that a point set be a simple closed curve. He found that, in addition to the simple closed curve being the common boundary of two domains (Jordan Curve Theorem), it is also accessible from each of its complementary domains, and this condition proved sufficient for a converse theorem. The set-theorist has taken the attitude that in getting a converse in three dimensions for the simple closed surface (H_2), one should look for a stronger kind of accessibility. Up to date such a procedure has not been found successful,† and I do not feel as I did formerly that such a procedure is really following out the spirit of Schoenflies' ideas. In view of the fact that he himself suggests the necessity of having regard for the higher connectivity of three-dimensional space (see quotation in §3), in extending his results to higher dimensions, it certainly seems to be appropriate to incorporate such information as is contained in the Alexander Duality Theorem in a converse of the theorem concerning the separation of E_3 by

* As shown by J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8–10.

† A special case of the converse, employing a *conical* accessibility, has been announced by T. C. Benton, this Bulletin, vol. 37 (1931), p. 34, abstract No. 80.

a simple closed surface (see last paragraph of §9). By following such a procedure, I have been able to get not only such a converse theorem, but have also found* that it is a special case of the converse of the theorem concerning the separation of E_3 by a closed two-dimensional manifold of genus k. This converse may be stated as follows. In E_3, let K be a closed and bounded point set such that (1) the Betti number $p^0(E_3-K)\geqq 1$, and the Betti number $p^1(E_3-K)$ is finite; and (2) if D is a domain complementary to K, then D is uniformly locally connected and every point of K is a limit point of D. Then K is a closed two-dimensional manifold of genus $k=\frac{1}{2}p^1(E_3-K)$. By setting $p^1(E_3-K)=0$ in this theorem we of course get the converse of the theorem regarding the separation of E_3 by a simple closed surface.†

No converse theorem for H_{n-1} in E_n, where $n>3$, has as yet been found. Apparently the converse announced by Rey Pastor recently‡ cannot be true, even in E_3, in view of the example of Alexander cited above.

Another fundamental result of Schoenflies is that a continuum K in E_2, the diameters of whose complementary domains form a zero sequence and the boundaries of whose complementary domains are accessible from all sides, is a Jordan continuum. R. L. Moore has shown by examples§ that this theorem is not

* See this Bulletin, vol. 36 (1930), p. 219, abstract No. 196; for even weaker conditions, see ibid., vol. 37, p. 519, abstract No. 236.

† See also my paper *A converse of the Jordan-Brouwer* · · · , loc. cit. I have recently come across an early attempt at the converse, for the case of an H_2 in E_3, in a paper by K. Kaluzsay, *A felületre vonatkozó Jordan-tétel megfordítása*, Mathematikai és Physikai Lapok, vol. 24 (1915), pp. 101–141. Upon obtaining a translation of Kaluzsay's results, I have found the interesting fact that his conditions (No. 5 apparently gives the uniform connectedness im kleinen of the complementary domains) closely approximate those which I gave in the Transactions paper just cited, except that instead of the condition on the Betti number which I used, he assumed that (condition 3) any closed polygon in a complementary domain can be continuously deformed into a point in that domain, thus yielding only a special case.

‡ J. Rey Pastor, *Une propriété caractéristique des variétés de Jordan*, Comptes Rendus, vol. 192 (1931), pp. 27–29. For further commentary on this paper, see L. E. J. Brouwer, *Über freie Umschliessungen im Räume*, Proceedings of the Royal Academy of Amsterdam, vol. 34 (1931), pp. 100–101.

§ *On the relation of a continuous curve to its complementary domains in space of three dimensions*, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 33–38.

true as it stands in E_3, and a characterization of Jordan continua by their external properties in E_3 is lacking. The chief reason for this is undoubtedly that not enough of the external properties have as yet been found, and some of these at least will be of a combinatorial nature. The most natural approach to the problem would probably be to consider first those Jordan continua that are the common boundaries of just two complementary domains. It is known that in such a case the possession of the property of being uniformly locally connected by the two domains is sufficient to make the common boundary a Jordan continuum,* but this is not a necessary condition.

To illustrate the complexities of the problem: In E_2 a Jordan continuum which is the common boundary of two domains is a simple closed curve (H_1). I have found† an example, however, in E_3, of a Jordan continuum which is the common boundary of three (any finite number, or a denumerable infinity of) domains, all of which are uniformly locally connected. On the other hand, I have found that if K is the common boundary of two uniformly locally connected domains D_1 and D_2 in E_3, and there exists a point P of K and a positive number ϵ such that all 1-cycles of $D_i \cdot S(P, \epsilon)$, $(i=1, 2)$, are homologous to zero in D_i, then K is the common boundary of only two domains. Thus, to be a common boundary of at least three such domains, a Jordan continuum must be linked by 1-cycles *in every neighborhood of every point of* the continuum. It would seem that if we do not allow a Jordan continuum which is a common boundary of two domains to be linked at all by cycles of dimension greater than zero, it is probably very greatly restricted. In particular, I have found that if a Jordan continuum which is the boundary of a domain D in E_n has no cut point (such is always the case when it is the *common* boundary of two domains), then, if the $(n-2)$-cycles of D all bound in D, they bound *uniformly* in D.

Before leaving this matter of the external properties of Jordan continua in E_n, I want to emphasize two matters. First, that *the connectedness of a domain is a special case of the bounding properties of its i-cycles*; in this sense the result stated at the end of the preceding paragraph is the extension to E_n of the uni-

* See R. L. Moore, *On the relation . . .*, loc. cit., Theorem 1, and my paper *A converse of the Jordan-Brouwer . . .*, loc. cit., p. 644, II.

† See this Bulletin, vol. 37 (1931), p. 525, abstract No. 258.

formly locally connected property of the domains bounded by a simple closed curve in E_2 (see third paragraph of §9). Second, *separation of E_n by a closed set of points K is equivalent to the linking of 0-cycles of the complement, and hence a special case of the linking of the set by i-cycles of its complement.* By keeping these two matters in mind, many plane theorems can be extended to higher dimensions. To illustrate the second, consider the Moore-Kuratowski theorem to the effect that if a Jordan continuum separates two points P and Q in E_2, then it contains a simple closed curve which separates them. That this theorem does not generalize to three dimensions as a *separation* theorem is well known. However, as a *linking* theorem it generalizes* to any number of dimensions as follows. If, in E_n, an $(n-2)$-cycle links a Jordan continuum M, then it links a simple closed curve in M. Furthermore, in connection with a result of Moore† to the effect that in E_2 the set of all points on all simple closed curves of a Jordan continuum M that enclose a given point P is the sum of a finite number of Jordan continua, we have, in E_n, the result that the set of all points that lie on simple closed curves of a Jordan continuum M that are linked by an $(n-2)$-cycle is the sum of a finite number of Jordan continua each of which consists of a finite set of true cyclic elements of M.

So far we have considered only external properties of Jordan continua, and I hope I have made it clear that by taking a combinatorial point of view in the complement of the continuum (all of the above results were obtained by using the set-theoretic method in the continuum), the prospect of extending theorems for the plane to higher dimensions and of determining external properties of Jordan continua, is very hopeful.

Let us now consider the internal structure of Jordan continua. The modern Polish and American set-theorists have used the set-theoretic method exclusively in investigating internal properties, the majority of their results extending to general spaces since the continuum is usually considered as a space in itself without reference to any imbedding space. The culmination of

* See my paper *On the linking of Jordan continua in E_n by $(n-2)$-cycles*, to appear soon in the Annals of Mathematics.

† R. L. Moore, *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 745–753, Theorems 7 and 8.

the attempt to divide the Jordan continuum into elements is the cyclic element theory of Whyburn,* which furnishes what we might call the *atomic* structure of the Jordan continuum in terms of which it takes on the semblance of an acyclic Jordan continuum. However, in higher dimensions the cyclic element, which is itself a Jordan continuum, may have a highly complicated structure, and the question arises as to whether there may not be devised some sort of *electronic* structure for it. Since the basis for the true cyclic element is the simple closed curve, or 1-cycle, I would suggest that we ought also to investigate the structure of the Jordan continuum with reference to its cycles of higher dimension. Of course I am not referring here to the geometric cycle of the combinatorial topology; rather am I referring to the *complete cycles* introduced by Vietoris,† which are based on the pure combinatorial notion of cycle. First, however, let me give a short exposition of this theory.

Let M be any closed subset of E_n; a set of $k+1$ points of M is called a *δ-simplex* of M of dimension k if the diameter of the set is $<\delta$. (Singularities may occur when points fall together.) The notion of a complex made up of such simplexes, called a *δ-complex*, of the *boundary* of such a complex, of *δ-cycles*, bounding relations, etc., are set up as usual. A δ-cycle of M is called *ϵ-homologous to zero in M* if it is the boundary of an ϵ-complex in M. We then may set up the notion of ϵ-independence of δ-cycles with respect to homologies, and the corresponding Betti numbers.

If ϵ_i, $(i=1, 2, 3, \cdots)$, is a sequence of positive numbers such that $\lim \epsilon_i=0$, and $r=(r_1, r_2, r_3, \cdots)$ is a sequence of r-dimensional ϵ_i-cycles, r_i, in M, then r is called an *r-dimensional complete cycle* of M if for every positive number ϵ there exists a number N such that for $k>N$, $\epsilon_k<\epsilon$, while for every two numbers k_1 and k_2 which are greater than N,

$$r_{k_1} \overset{\epsilon}{\sim} r_{k_2}, \qquad \text{in } M.$$

Two complete cycles r and r' are called homologous if for every positive number ϵ there exists an N such that for all $k>N$,

* See C. Kuratowski and G. T. Whyburn, *Sur les éléments cycliques et leurs applications*, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

† See Vietoris, loc. cit., and Alexandroff, *Gestalt und Lage*, loc. cit.

$$r_k \overset{\epsilon}{\sim} r_k', \qquad \text{in } M.$$

Now, as to the relation between the closed subsets of M, and these complete cycles and their homologies: If F is a closed subset of M and r is a complete cycle of M, then, if r is also a complete cycle of F, F is called a *carrier** of r. The common part of all carriers of r is also a carrier of r and is called the *smallest carrier* of r. We introduce also the notion of *carrier of a homology*. A complete cycle r is homologous to zero in a closed set F if each r_i bounds an η_i-complex K_i in F such that $\lim \eta_i = 0$. Every subset F' of F which is a topological limit of a convergent subsequence† of such sets K_i is called a carrier of the homology $r \sim 0$ in M.

Let us call a complete cycle *essential* if it has at least one carrier in which it is not homologous to zero. Then, for instance, every simple closed curve of a Jordan continuum M is the carrier of an essential 1-dimensional complete cycle C^1 of M. If $C^1 \sim 0$ in M, every carrier of this homology contains an *irreducible membrane*‡ with respect to C_1; that is, a closed subset of M which is irreducible with respect to the property of being a carrier of this homology. This irreducible membrane cannot be disconnected by the omission of an at most 0-dimensional subset, and if it is not of higher dimension than 2, is a *2-dimensional cantorian manifold*.§

We can also introduce the notion of two simple closed curves J and K of a Jordan continuum M being *equivalent*, or homologous to one another, if complete cycles of which they are the respective carriers are homologous to one another in M.

In like manner we can consider the higher dimensional complete cycles of M and their carriers. I believe that a consideration of these cycles, related cantorian manifolds, and of the linkings by cycles of the complement of M, might lead to a considerable theory of Jordan continua in higher dimensions.

Aside from the theory of Jordan continua, the irreducible membrane, being a generalization of the irreducible continuum,

* *Träger*; see *Gestalt and Lage*, pp. 168–169.

† See Hausdorff, *Mengenlehre*, p. 146.

‡ See *Gestalt und Lage*, p. 179.

§ As defined by Urysohn, an n-dimensional cantorian manifold is an n-dimensional closed point set which is not disconnected by the omission of any of its at most $(n-2)$-dimensional closed subsets.

offers great possibility of investigation of structural properties by the unified method.*

11. *Continua.* Concerning the internal structure of continua that are not necessarily Jordan continua we find a notable collection of papers in the set-theoretic topology. For instance a decomposition into elements, such as the upper semi-continuous collections of continua of Moore, similar to the decomposition of the Jordan continuum into cyclic elements, has been studied by several authors. Limitations of space forbid my giving an exposition of what has been done along this line.† Since, in taking up properties of continua that are imbedded in E_n, I shall wish to make use of certain combinatorial properties of general closed sets, it will be convenient to go directly to the latter.

12. *Closed Sets in General.* The most important result which has recently been obtained I believe to be a duality theorem, proved independently by Frankl, Alexandroff, and Lefschetz.‡

Let M be any closed subset of H_n. We then define complete cycles in M as above. A system of complete cycles of M,

$$r_1, r_2, r_3, \cdots, r_m, \cdots, \tag{1}$$

is called a *basis for the r-dimensional complete cycles of M* if the cycles (1) are linearly independent in the sense that no finite linear combination of them is homologous to zero in M, and if for every ϵ there is a finite subsystem (1′) of (1) such that every

* See *Gestalt und Lage*, pp. 179–181.

† I believe that Moore's recent colloquium lectures on point set theory treat of decompositions of continua rather extensively. It may be of interest to call attention here to results of W. Hurewicz, *Über oberhalbstetige Zerlegungen von Punktmengen in Kontinua*, Fundamenta Mathematicae, vol. 15 (1930), pp. 57–60, to the effect that an upper-semicontinuous decomposition of a closed set in E_2 constitutes an at most 2-dimensional space, whereas in E_3 there exists a curve M such that for every compact space S there is a subset M_S of M and an upper-semicontinuous decomposition of M_S which is homeomorphic with S.

‡ F. Frankl, *Topologische Beziehungen in sich kompakter Teilmengen euklidischen Räume zu ihren Komplementen sowie Anwendung auf die Prim-Enden-Theorie*, Wiener Akademie der Wissenschaften, Math.-Naturw. Kl., Sitzungsberichte, Abt. 2A, vol. 136 (1927), pp. 689–699; *Gestalt und Lage*, pp. 156 ff. (also see Göttinger Nachrichten, Math.-Phys. Kl., Nov. 25, 1927); S. Lefschetz, *Closed point sets on a manifold*, Annals of Mathematics, vol. 29 (1928), pp. 232–254, as well as earlier papers referred to therein.

r-dimensional complete cycle is ϵ-homologous to a linear combination of (1').* The number of cycles in (1) is a topological invariant of M, and may be called the *r-dimensional Brouwer number* (since for the case $n=2$ and $r=1$ it was originally set up by Brouwer) or the *r-dimensional Betti number of M*. I shall use the latter terminology.†

A system of $(n-r-1)$-dimensional cycles (geometric) of H_n-M

$$\Gamma_1^{n-r-1}, \Gamma_2^{n-r-1}, \cdots, \Gamma_m^{n-r-1}, \cdots, \tag{2}$$

is called an $(n-r-1)$-dimensional basis for H_n-M if no finite linear combination of them is homologous to zero in H_n-M and every $(n-r-1)$-dimensional cycle of H_n-M is homologous to a finite linear combination of them.

Let r be an r-dimensional complete cycle in M, and Γ^{n-r-1} an $(n-r-1)$-cycle of H_n-M. Then r and Γ^{n-r-1} will be said to be linked if Γ^{n-r-1} links almost all of the ϵ_i-cycles that make up r, when these are geometrically realized. Then the bases (1) and (2) may be so chosen that their cycles are uniquely linked; that is, r_m and Γ_m^{n-r-1} are linked, but r_m and Γ_i^{n-r-1} for $i \neq m$ are not linked. Consequently, if, in the Alexander Duality Theorem, M stands for a closed set of points, the theorem still holds. And as the Betti number of M is an invariant, *the number* $p^{n-r-1}(H_n-M)$ *is an invariant*. One of the most important consequences of this is obtained for the case $r=n-1$, in which we obtain the result that *the number of domains complementary to a closed set of points is invariant*. Here we have again an extension to n dimensions of an important theorem of plane topology, this theorem having been proved for the number of domains complementary to a closed set in E_2 by Brouwer in 1912.‡

In case a closed set M separates two points P and Q of E_n,

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* See *Gestalt und Lage*, pp. 159–160, and Vietoris, loc. cit., part II.

† Obviously $p^0(M)+1$ is the number of components of M. It may be of interest to point out here a relation between this part of the theory of complete cycles and the point set theory of Cantor, namely, that the Cantor definition of connectedness, defined in terms of "ϵ-chains," demanded, if translated into combinatorial terminology, that the 0-dimensional complete cycles of M should all be homologous to zero in M.

‡ *Beweis der Invarianz der geschlossenen Kurve*, loc. cit.

then, as has been shown by Mazurkiewicz,* M contains a set F which is an *irreducible cut* between P and Q. Since a common boundary of two domains in E_n is necessarily an irreducible cut between points in these respective domains, it follows from Mazurkiewicz' result that a necessary and sufficient condition that a set M in E_n be an irreducible cut between some two points is that it be the common boundary of two domains. That an irreducible cut between two points is a continuum follows from the Phragmen-Brouwer Theorem, about which we shall speak below when we consider separation theorems.

As separation of E_n is equivalent to linking by 0-cycles, the result just stated may be formulated as follows. In order that a set M in E_n should be irreducibly linked with a 0-cycle of $E_n - M$ it is necessary and sufficient that M be the common boundary of two domains. The question arises, is the property of a set M being the common boundary of two domains invariant with the set M? And can a necessary and sufficient condition for a set M being irreducibly linked with an r-cycle of $E_n - M$, where r is not necessarily equal to 0, be obtained? By making use of the theory of the complete cycles of a closed set, Alexandroff has obtained† the answers to these questions, by getting the necessary and sufficient condition asked for, and from this deducing that the existence of an r-cycle irreducibly linked with M is topologically invariant with M; as a particular case of this, the property of being a common boundary of two domains of E_n is a topological invariant of M.

However, a set M can be an irreducible cut between two points without being a *completely irreducible cut*‡ of E_n. For the case of E_2, Kuratowski has shown§ that if a set M which cuts the plane has only a finite number of complementary domains,

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* Fundamenta Mathematicae, vol. 1 (1920), p. 63. A set F is an *irreducible cut* of E_n between P and Q if it separates P and Q but no proper closed subset of F separates P and Q. The generalization of this notion to that of a set F which is *irreducibly linked* by a cycle of $E_n - F$ is obvious, separation being a special case of linking (see §10).

† *Gestalt und Lage*, pp. 169 ff.

‡ A cut F is a *completely irreducible cut* of E_n if no proper closed subset of it separates points of E_n. Thus, a simple closed curve in E_2 is a completely irreducible cut of E_2.

§ C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6 (1924), pp. 130–145.

then it contains a completely irreducible cut of the plane, and points out that if the number of domains is infinite, there may be no such cut. However, if M is a Jordan continuum that cuts the plane, it does contain a completely irreducible cut, namely an H_1. In the case of E_n, it is still true (following Kuratowski's proof, which depends only on Mazurkiewicz' theorem) that if M has only a finite number of complementary domains, it contains a completely irreducible cut of E_n. In view of what I have said in §10 concerning separation of E_n being a special case of linking, problems arise as to the extension of these theorems to theorems concerning the linking of a set by r-cycles, $r>0$.

Let us consider, first, the notion of a *closed m-dimensional cantorian manifold*, introduced by Alexandroff.* A closed set of points of dimension m whose mth Betti number is not zero, while the mth Betti number of each of its proper closed subsets is zero, is called a closed m-dimensional cantorian manifold. By virtue of the invariance of the Betti numbers, the property of being a closed m-dimensional cantorian manifold is a topological invariant. And in view of the duality theorem mentioned above, the completely irreducible cuts of E_n are identical with the closed $(n-1)$-dimensional cantorian manifolds of E_n, and thus are identical with those sets that separate E_n and are the common boundaries of all their complementary domains. Thus we have a characterization of these manifolds by their external properties, something we lack for the combinatorial manifolds, excepting, of course, the case of the 1-manifold in E_2 (Schoenflies' converse of the Jordan Curve Theorem) and the 2-manifold in E_3 (see §10).

To return to the problems mentioned above: It follows at once, from an induction theorem of Brouwer,† that if a closed set M is linked by an r-cycle, then it contains a set F which is irreducibly linked by this r-cycle. Regarding the generalization of the theorem of Kuratowski regarding completely irreducible cuts of E_2, I have shown that if a closed set M is r-dimensional and is linked by only a finite number of $(n-r-1)$-cycles, then it contains a closed r-dimensional cantorian manifold. As a special

* *Gestalt und Lage*, p. 176.

† L. E. J. Brouwer, *On the structure of perfect sets of points*, Proceedings of the Amsterdam Academy, vol. 14 (1912), pp. 137–147.

case of this theorem it follows that if a set separates E_n into a finite number of domains, then it contains a completely irreducible cut of E_n (the hypothesis as to the dimensionality of M being superfluous in this case). The problem as to whether the restriction that M be linked by only a finite number of $(n-r-1)$-cycles is necessary in case M is a Jordan continuum remains open, except, of course, for the case $r=1$ (see §10).

The case of a bounded set C which is the *common boundary* of two (or more) domains in E_n was considered by Urysohn in his original memoirs on dimension theory. Urysohn showed that, in E_3, the set C is a 2-dimensional cantorian manifold and indicated a general plan for proof of the fact that, in E_n, the set C is an $(n-1)$-dimensional cantorian manifold. A complete proof of the latter fact was later given by Alexandroff, who furthermore showed that if K is a closed subset of C such that $p^{n-2}(K)=0$, then $C-K$ is connected.* I have recently proved a theorem which contains both of these results as special cases, namely, that if $p^{n-2}(K)=k$, then $C-K$ has at most $k+1$ components; and if C is a common boundary of at least three domains, then $C-K$ has at most k components. In this same connection I have also shown that if M is any closed m-dimensional cantorian manifold and K a closed subset of M such that $p^{m-1}(K)=k$, then $M-K$ has at most $k+1$ components; and that if $p^m(M)>1$ and $p^{m-1}(K)>0$, then $M-K$ has at most k components.

In regard to those closed sets that are characterized only by their being boundaries of domains, important results are embodied in the extension by Kaufmann† of the Carathéodory theory of prime ends to E_3. The details of this are too long to include here. Instead of four types of prime ends, as set up by Carathéodory for E_2, Kaufmann distinguishes six types in E_3.

The accessibility of general closed sets in E_n has been studied to a certain extent. I have spoken above of the accessibility theorems obtained by Mazurkiewicz and myself. In general, it

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* *Gestalt und Lage*, p. 154.

† Boris Kaufmann, *Über die Berandung ebener und raumlicher Gebiete (Primendentheorie)*, Mathematische Annalen, vol. 103 (1930), pp. 70–144. See also H. Teresaka, *On the boundary of open surface*, Japanese Journal of Mathematics, vol. 8 (1931), pp. 49–64.

is well known, as a result of a standardized procedure for proving accessibility, that a set M which does not separate any domain of E_n is accessible at all its points. Thus, since Urysohn showed* that no F_σ which is at most $(n-2)$-dimensional cuts a domain, it follows that every point of such a set is accessible from its complement. In particular, every at most $(n-2)$-dimensional closed set is accessible from its complement.

Further results on closed sets will be mentioned below in other connections.

13. *Separation Theorems.*† From a set-theoretic point of view, these were reported upon by Kline. I wish, however, to point out an important set of *addition* theorems, obtained by the combinatorial method, which are fundamental in the investigation of separation (and linking) properties of closed sets in higher dimensions.

The model for all later addition theorems is that of Alexander,‡ which states that if M is the sum of two closed sets A_1 and A_2 in E_n, and Γ^i an i-cycle of E_n-M such that there is an $(i+1)$-complex L_j^{i+1} bounded by Γ^i in E_n-A_j $(j=1, 2)$, and such that the cycle $L_1^{i+1}+L_2^{i+1}$ bounds in $E_n-A_1\cdot A_2$, then the cycle $\Gamma^i\sim 0$ in E_n-M. This is called an *addition theorem* since non-linking by each of two sets is extended, under the given assumptions, to non-linking by their sum. The set-theorist will recognize immediately that the classical Phragmen-Brouwer theorem, which states that if neither of two mutually exclusive bounded closed sets separates the plane, their sum does not, follows at once, and for E_n, from this theorem—a fact which probably led Alexandroff to apply it to obtain vast generalizations of the Phragmen-Brouwer Theorem for E_n.

From the theorem just stated, and from the duality theorem for closed sets, it follows that if A_1 and A_2 are closed sets in E_n, and a cycle $\Gamma^i\sim 0$ in both E_n-A_1 and E_n-A_2, and $p^{n-i-2}(A_1\cdot A_2)=0$, then Γ^i does not link the sum A_1+A_2. Moreover, Alexandroff shows§ that if A_1 and A_2 are closed subsets of E_n,

* Fundamenta Mathematicae, vol. 8 (1926), p. 355.

† Obviously several of the results just stated in §12 might properly come under this heading.

‡ *A proof and extension of the Jordan-Brouwer separation theorem*, loc. cit., Corollary W^i.

§ *Gestalt und Lage*, p. 178.

and Γ^i a cycle of $E_n-(A_1+A_2)$ which links A_1+A_2 but not A_1 or A_2, then there exists a complete $(n-i-2)$-cycle in $A_1 \cdot A_2$ which is homologous to zero in both A_1 and A_2 but is not homologous to zero in $A_1 \cdot A_2$.

The epitome of this order of ideas is contained in two theorems of Alexandroff, the first related to the theorem just stated. If *every* cycle $\Gamma^i \sim 0$ in both E_n-A_1 and E_n-A_2, then there exists a cycle K^i which links A_1+A_2 if and only if there exists an $(n-i-2)$-complete cycle in $A_1 \cdot A_2$ which is homologous to zero in A_1 and A_2, but not in $A_1 \cdot A_2$. Just as the classical Phragmen-Brouwer theorem follows from the addition theorem of Alexander, so follows from the theorem just stated the following generalization of the Phragmen-Brouwer theorem. If neither of two closed sets A_1, A_2 separates E_n, then their sum separates E_n if and only if there exists an $(n-2)$-complete cycle in $A_1 \cdot A_2$ which is homologous to zero in both A_1 and A_2, but not in $A_1 \cdot A_2$. Thus we have a necessary and sufficient condition that the sum of two closed sets, neither of which separates space, may have a sum that separates space. In this connection it is interesting to note that Kline, in the concluding remarks of his 1927 lecture in New York, gave special mention to the problem of finding necessary and sufficient conditions that the sum of two closed *connected* sets, neither of which separates *three*-space, may have a sum that separates three-space. A special case of Alexandroff's theorem is of course a solution of this problem.

I think that sufficient has already been said to indicate the importance of addition theorems in problems concerning the separation, or, in more general terms, the *linking*, of closed point sets in E_n. Theorems which concern only the addition of two sets seem to have been sufficient for the treatment of a large number of problems. It is natural, however, to expect that the extension of these investigations to cases of more than two sets may prove valuable, and I find that E. Čech has recently considered this matter. His results* involve too many details to allow of their inclusion here, but he has obtained, for instance, conditions sufficient that a cycle Γ^i complementary to all of k sets, $A_1, A_2, \cdots, A_k$, and not linking any $k-1$ of these sets, should not link their sum (as a matter of fact, Čech considers

* E. Čech, *Trois théorèmes sur l'homologie*, Publications de la Faculté des Sciences de l'Université Masaryk, 1931, Čis. 144.

the more general case of s cycles, where s is a positive integer, and the condition required is that a certain non-vacuous sum of these shall not link the sum of the given sets).

14. *Imbedding Problems.* (1) When, in general, can a space R be imbedded in a euclidean space? (2) Under what conditions can a set be imbedded in a Jordan continuum?

The results that have been obtained in the past few years have, I believe, enhanced considerably the importance of both euclidean spaces and Jordan continua. In general, we say that a space R can be imbedded in a space S, when there exists a subset R' of S whose neighborhoods are defined by its overlappings with neighborhoods of S, and such that R and R' are homeomorphic. For example, every n-dimensional compact metric space can be imbedded in a euclidean space (see below), so that the study of such spaces is identical with the study of bounded closed subsets of euclidean space; in addition, if it is of positive dimension, such a space can be imbedded in a Jordan continuum of the same dimension.

(1) Of central importance among the general topological spaces are the metric separable and the compact metric spaces. That the degree of generality in these conceptions is not as great as might at first be supposed was made evident by Urysohn's discovery that every separable metric space is imbeddable in the Hilbert parallelotope.* That a separable metric space of finite dimension is imbeddable in a compact metric space of the same dimension was proved by Hurewicz.†

That every compact metric space of n dimensions can be imbedded in E_{2n+1} was shown by Menger‡ and in combination with the result of the preceding paragraph we have a similar theorem for a separable metric space. More recent proofs of the Menger imbedding theorem have been given by Nöbeling, Lefschetz, and by Pontrjagin and Tolstowa.§

* P. Urysohn, *Der Hilbertsche Raum als Urbild der metrischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 302–304.

† Hurewicz's first proof was apparently unsatisfactory—see Menger, *Dimensionstheorie*, remarks bottom p. 294; however, see W. Hurewicz, *Über Einbettung separabler Räume in gleichdimensionale kompakte Räume*, Monatshefte für Mathematik und Physik, vol. 37 (1930), pp. 199–208.

‡ See *Dimensionstheorie*, p. 295.

§ G. Nöbeling, *Über eine n-dimensionale Universalmenge im* R_{2n+1}, Mathematische Annalen, vol. 104 (1930), pp. 71–80, S. Lefschetz, *On compact spaces*,

In his book, Menger sketched a proof of the theorem that every n-dimensional separable metric space is homeomorphic with a subset of E_{2n+1}, and asserted* that there exists for every n a universal n-dimensional separable metric space. The proof of this assertion has been provided by Lefschetz and Nöbeling.† Lefschetz first obtains a universal compact metric space by employing the product of a certain series of $(2n+1)$-dimensional simplexes in E_{2n+1}, and then applying the imbedding theorem for separable spaces. Nöbeling shows that the set of all points of E_{2n+1} with at most n rational coordinates is an n-dimensional universal separable metric space, thus obtaining an existence proof for the universal space and a proof of the imbedding theorem for separable spaces to euclidean spaces in one.

Regarding further results concerning the existence of universal imbedding spaces, special results concerning possibility of imbedding in E_2, etc., space limitations prevent my going into detail. There remain many unsolved problems concerning imbedding in E_2 and E_3, in particular.‡

(2) The complete answer to the second question proposed above has been given by Alexandroff and Tumarkin who showed that every compact metric space of positive dimension can be imbedded in a Jordan continuum of the same dimension.§

15. *Imbedding by Extension.* Given a set of points M in a space S, when can it be imbedded *by extension* in a set N of a given type T; that is, when does there exist a set of points K in $S-M$ such that $M+K$ is of type T?

Annals of Mathematics, vol. 32 (1931), pp. 521–538, L. Pontrjagin and G. Tolstowa, *Beweis des Mengerschen Einbettungssatzes*, Mathematische Annalen, vol. 105, pp. 734–745.

* *Dimensionstheorie*, p. 314.

† Loc. cit.

‡ Conditions necessary and sufficient that a separable metric space be imbeddable in E_1 have been given by L. W. Cohen, *A characterization of those subsets of metric separable space which are homeomorphic with subsets of the linear continuum*, Fundamenta Mathematicae, vol. 14 (1929), pp. 281–303.

§ P. Alexandroff and L. Tumarkin, *Beweis des Satzes, dass jede abgeschlossene Menge positiver Dimension in einem lokal zusammenhängenden Kontinuum von derselben Dimension topologisch enthalten ist*, Fundamenta Mathematicae, vol. 11 (1928), pp. 141–144. For the zero-dimensional case see W. Sierpinski, *Sur les ensembles connexes et non connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 81–95, and Cohen, loc. cit.

If M is a closed, compact, totally disconnected point set, S is E_n, and N is an H_{n-1}, then the imbedding by extension is always possible, as shown by Antoine.*

For the case where S is E_n and N is an arc, the problem has been solved by Moore and Kline† ($n=2$) and E. W. Miller‡ ($n>2$), by the furnishing of necessary and sufficient conditions for M. For the case where S is E_n and N is a Jordan continuum, Gehman showed§ that every bounded continuum M in E_2 is a subset of a Jordan continuum obtained by adding to M a denumerable infinity of arcs, and Whyburn and Ayres‖ extended this result to the case where S is a Jordan continuum. In case S is any open connected subset of a Jordan continuum, and M is a compact closed subset of S, I have shown that there exists in S a Jordan continuum containing M.¶ For the case where S is any metric space, I have shown** that in order that a compact closed subset of S should be a subset of a Jordan continuum obtained by adding a denumerable infinity of arcs of S to M, it is necessary and sufficient that M should be arcwise connected through S and arcwise connected im kleinen through S, thus bringing the problem for the case where N is a Jordan continuum and S any metric space to its completion. Due to the nature of the problem (adjustment of a local property, that is,

* L. Antoine, *Sur les ensembles parfaits partout discontinus*, Comptes Rendus, vol. 173 (1921), pp. 284–285. See also, by the same author, *Sur l'homéomorphie de deux figures et de leurs voisinages*, Journal de Mathématiques, vol. 4 (1921), pp. 221–325, and *Sur les voisinages de deux figures homéomorphes*, Fundamenta Mathematicae, vol. 5 (1924), pp. 265–287. Also S. Saks, *Sur l'homéomorphie des variétés à deux dimensions*, Fundamenta Mathematicae, vol. 5 (1924), pp. 288–320.

† R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Annals of Mathematics, vol. 20 (1919), pp. 218–223.

‡ E. W. Miller, *On subsets of a continuous curve which lie on an arc of the continuous curve*, American Journal of Mathematics, vol. 54 (1932), pp. 397–416.

§ H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem 2.

‖ G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, this Bulletin, vol. 34 (1928), pp. 349–360, Theorem 1.

¶ R. L. Wilder, *On connected and regular point sets*, this Bulletin, vol. 34 (1928), pp. 649–655, Theorem 5.

** R. L. Wilder, *On the imbedding of subsets of a metric space in Jordan continua*, Fundamenta Mathematicae, vol. 19 (1932), pp. 45–64.

local connectedness), these results have all been obtained by the set-theoretic method.*

It need hardly be indicated that there are many open problems concerning this matter of imbedding by extension. Thus, in E_n every arc is a subset of a simple closed curve (because of the accessibility of its end points). However, it is easy to show that not every 2-cell imbedded in E_3 is a subset of a 2-sphere imbedded in E_3.† It might be interesting to determine conditions under which such an extension of a 2-cell may be carried out. Probably the deformability of cycles of the space complementary to the 2-cell, as well as the higher connectivity of the complementary space, must be taken into account here.‡

16. *Dimension Theory.* This theory, in general considered a special province of the set-theoretic topology, has been treated by Menger in his book and by others in several reports.§ That it is, however, amenable to combinatorial attack, has been forecast by several results. To quote Lefschetz,‖ "The unavoidable link between dimensionality and combinatorial topology is the Lebesgue-Urysohn-Menger theorem on the order of covering sets." According to an early result of Alexandroff,¶ the dimen-

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* Imbedding theorems of a more special type have been given by C. M. Cleveland, *On the existence of acyclic curves satisfying certain conditions with respect to a given continuous curve*, Transactions of this Society, vol. 33 (1931), pp. 958–978, and Zippin, *Generalization of a theorem due to C. M. Cleveland*, American Journal of Mathematics, vol. 54 (1932), pp. 176–184. Also see W. Stepanoff and W. Tumarkin, *Über eine Erweiterung abgeschlossenen Mengen zu Jordanschen Kontinuen derselben Dimension*, Fundamenta Mathematicae, vol. 12 (1928), pp. 43–46, and G. T. Whyburn, *On the set of all cut points of a continuous curve*, ibid., vol. 15 (1930), pp. 185–194.

† By the method used by Alexander (*An example of a simply connected surface . . .*, loc. cit.). Thus, let T be a square plus its interior, and begin attaching the surfaces S_i of Alexander at two distinct non-overlapping circular regions of T.

‡ The same remark applies to the problem of extending the homeomorphism between two simple closed surfaces to the entire space. See end of §9.

§ For recent results, G. Nöbeling, *Die neuesten Ergebnisse der Dimensionstheorie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 41 (1931), pp. 1–17.

‖ S. Lefschetz, *On compact spaces*, loc. cit.

¶ P. Alexandroff, *Sur la dimension des ensembles fermés*, Comptes Rendus de l'Académie des Sciences, vol. 183 (1926), pp. 640–642. See also references to Brouwer therein.

sion of a closed set F is the smallest number m having the property that for every $\epsilon>0$, F is ϵ-deformable into a complex of dimension m, thus relating the dimensionality of closed point sets to the dimensionality of ordinary complexes.

Just as in the case of the theory of measure, in setting up a theory of dimension one looks for a theory which will accomplish certain objects intuitively demanded, such as, for instance, that E_n shall be, according to the theory proposed, n-dimensional. This leads to the main problems of the dimension theory, namely, the setting up of various *Rechtfertigungssätze*. In the Menger-Urysohn theory most of the desired theorems of this type were established early in the development of the theory, excepting in the case of the question as to whether every $(n-1)$-dimensional closed subset of E_n separates some domain, and the question as to whether the product theorem holds. The former question has recently been answered affirmatively by Frankl, Pontrjagin and Alexandroff,* and since, as shown by Urysohn, no closed subset of E_n of lower dimension separates any domain, we now have a characterization of $(n-1)$-dimensionality of closed sets in E_n by their external properties alone. This raises the question as to whether the same cannot be established for lower dimensionalities. Since the characterizing of $(n-1)$-dimensionality of a set F in E_n is accomplished by its *separation* properties, we should expect that the characterization of lower dimensionalities might be achieved by the *linking* properties of sets—we shall see below that this is exactly the case. That the product theorem† does not hold for the Menger-Urysohn dimension theory has been established lately by Pontrjagin,‡ who has demonstrated the existence, in E_4, of two closed sets F' and F'' such that $\dim F'=\dim F''=2$, while $\dim (F'\times F'')=3$. And here the question arises, can we not set up a dimension theory in which the product theorem holds, without sacrificing any of the other *Rechtfertigungssätze*?

* F. Frankl and L. Pontrjagin, *Ein Knotensatz mit Anwendung auf die Dimensionstheorie*, Mathematische Annalen, vol. 102 (1930), pp. 785–789; F. Frankl, *Charakterisierung der $(n-1)$-dimensionalen abgeschlossenen Mengen des R^n*, ibid., vol. 103 (1931), pp. 784–787; P. Alexandroff, *Analyse géométrique de la dimension des ensembles fermés*, Comptes Rendus, vol. 191 (1930), pp. 475–477.

† That is, $\dim (F_1\times F_2)=\dim F_1+\dim F_2$.

‡ L. Pontrjagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, Comptes Rendus, vol. 190 (1930), pp. 1105–1107.

The answer to the question, provided by a use of the theory of complete cycles from the standpoint of a unified analysis situs, will appear shortly in a memoir of Alexandroff.* I shall not go into complete details of Alexandroff's work, but will give enough of its characteristics and results to indicate the general direction in which it proceeds. A set F (hereafter F will always denote a closed set of points) is called r-dimensional if it has an essential $(r-1)$-dimensional complete cycle which is homologous to zero in F, but no essential complete cycle of higher dimension which is homologous to zero in F. According to the modulus used in the homologies, one of course defines thus the *dimension modulo* m, conveniently denoted by $\Delta^m(F)$. That this definition agrees with the ordinary intuitive notion of dimension is apparent from a consideration of euclidean elements. If one allows the modulus employed in the definition of r_k, where $r=(r_1, r_2, \cdots, r_k, \cdots)$ is the cycle employed in this definition, to vary with k, we get a complete cycle according to variable modulus, and a corresponding *dimension according to variable modulus*, which we denote by $\Delta(F)$; this turns out to be identical with the ordinary Brouwer-Menger-Urysohn dimension of closed sets, which is thus a special case of the modular dimensions.

For each of these modular dimensions a dimension theory may be developed. In each such theory we have the summation theorem: "An n-dimensional set F is not the sum of countably many $(n-1)$-dimensional closed sets"; the Brouwer invariance principle: "An n-dimensional F cannot be carried into a set of lower dimension by optionally small continuous deformations"; the theorem that "Every n-dimensional F contains an n-dimensional cantorian manifold"; and so on.

Since these dimension theories appear to parallel one another, may they not all be identical in the sense that they all yield the same dimension number for a given set F? It turns out that in E_3 they do yield the same dimension but in higher dimensions this is no longer the case, as has been shown by Pontrjagin.†

As to the *Rechtfertigungssatz* referred to above: One of the most interesting results I find in this work is the strong "*all-*

* *Dimensionstheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Mathematische Annalen, vol. 106 (1932), pp. 161–238; also earlier papers in Comptes Rendus referred to therein.

† *Sur une hypothèse* · · · , loc. cit.

gemeine dimensions-theoretische Rechtfertigungssatz". In E_n, if $\Delta(F)$ $(\Delta^m(F),\ m \geqq 2) = r$, then there exist spherical neighborhoods U_n of E_n in which there lie $(n-r-1)$-dimensional polyhedral cycles (cycles modulo m) which are linked with F in U_n; if, on the other hand, $\Delta(F)$ $(\Delta^m(F))$ is smaller than r, then every $(n-r-1)$-dimensional polyhedral cycle (cycle modulo m) complementary to F and lying in a spherical neighborhood U_n, bounds a complex in $U_n - F \cdot U_n$. Thus, in E_n, we are able, by the external properties of a closed set, to determine its dimension, either Menger-Urysohn or modulo $m (m \geqq 2)$.

For the product theorem, we have the interesting result found by Pontrjagin, that *if m is zero or a prime number, the dimension modulo m satisfies the product theorem.*

It appears, then, that the unified method not only makes an essential contribution to the theory of dimension, but opens up a much wider horizon here just as it does in the other aspects of point set theory discussed above. To quote from Alexandroff's introductory paragraphs:* "The actual application of the combinatorial notion of the homology to the general set-theoretic forms permits us to conclude that the dimension theory is not at all an isolated theory, but merely the first section of a general investigation, still in its infancy, of the bounding- and cut- (especially linking-) constructions in closed sets, and as such is to be developed · · · . The isolated position of the dimension theory was caused by the lack of a proper foundation in its set-theoretic development; this foundation, which will be given in the present work, indicates · · · the end of the self-sufficient existence of a theory and its entrance into a more universal chapter of mathematics." The challenge in these words is clear-cut; it remains only to be seen if it is heeded.

17. *Concluding Remarks.* I make no claim to the absolute completeness of the above report so far as the totality of results on higher dimensional point sets is concerned; thus, I have included nothing on the researches of Lefschetz on the fixed point problem, the work of Hopf in extending the theory of the *Abbildungsgrad* of Brouwer; to do all that might be done would necessitate more range than the scope of a symposium lecture allows. I hope, however, that enough has been said to

* *Dimensionstheorie*, loc. cit.

convince topologists that the true direction to the investigation of point sets in higher dimensions is indicated by the unified method, which takes into account not only the set-theoretic properties but the connectivity of the point sets as expressed in the Betti numbers. Although, as I have stated above, it is perhaps possible, without reference to the classical combinatorial topology, so to expand the set-theoretic method as to take account of this higher connectivity, I see no reason why we should not consider the theory of complete cycles as that expansion of the set-theoretic method which the set-theorists hope to find; what expression of the character of the higher connectivity can we desire, other than that expressed by connectivity numbers already known?

We have seen, in the above report, that many of the theorems of the set-theoretic topology concerning the separation of space are only special cases of the linking of combinatorial cycles of the complement of a point set. To ignore this fact is to struggle on blindly without a true perspective. And to attempt to build up a theory of higher dimensional point sets without taking into account higher connectivity is to refuse to explore the wider horizon that beckons beyond.

I consider the present situation of a division into two "schools" as the greatest menace to the future development of topology, a division which fosters a lack of true perspective, unscientific animosity, and an unnecessary delay in the progress of our investigations of the structure of space. If what I have said, even if it does not meet universal agreement, leads topologists to face the facts of the situation and hence to some remedy, be it that recommended here or some other, I shall be well satisfied. I hope I have presented evidence enough that the unified method (1) permits us to extend results to higher dimensions, where the properties in the large become complicated, (2) allows great generalization, even in the case of theorems which to the classical set-theoretic point of view appear incapable of generalization, and (3) opens up a vaster field of investigation. I hope that the opportunities for research, indeed, the necessity for research, in this new chapter of analysis situs, will no longer be neglected by the majority of topologists.

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AN INVERSIVE ALGORITHM*

BY D. H. LEHMER†

Viggo Brun‡ has given an algorithm for calculating directly the nth prime number from certain values of the function $\pi(x)$, the number of primes $\leqq x$. It is the purpose of this note to show that this algorithm is not peculiar to primes, but on the contrary, may be used to exhibit the nth member of any infinite class C of positive integers. With this degree of generality it is possible to use the algorithm to obtain identities between numerical functions.

The algorithm may be described as follows. Associated with the class C is the enumerative function $\theta(x)$ giving the number of members of C which are $\leqq x$. If n is any positive integer we form the sequence

$$n_0, n_1, n_2, \cdots, n_r, \cdots, \tag{1}$$

whose terms are defined as follows:

$$\begin{aligned} n_0 &= n, \\ n_1 &= n - \theta(n_0), \\ n_2 &= n - \theta(n_0 + n_1), \\ n_3 &= n - \theta(n_0 + n_1 + n_2), \\ &\cdots\cdots\cdots\cdots \\ n_r &= n - \theta(s_r), \end{aligned} \tag{2}$$

where, for brevity, we have written

$$s_r = n_0 + n_1 + n_2 + \cdots + n_{r-1}.$$

We have then the following theorem.

THEOREM. *The terms of the sequence* (1) *do not increase, and ultimately become and remain zero. If k is the rank of the first zero term of* (1), *then s_k is the nth member of the class C.*

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* Presented to the Society, August 31, 1932.

† National Research Fellow.

‡ Kongelige Norske Videnskaps-Selskabet, vol. 4, pp. 66–69.

PROOF. We first show that the terms of (1) are non-negative. This is true of $n_0=n$. If it is true of n_{r-1} we may show it true for n_r as follows. Consider the difference

$$(3) \quad n_{r-1} - n_r = \theta(s_r) - \theta(s_{r-1}) = \theta(s_{r-1} + n_{r-1}) - \theta(s_{r-1}).$$

If $n_{r-1}=0$, it follows that $n_r=0$. If $n_{r-1}>0$, (3) gives the number of members of C lying among the n_{r-1} consecutive integers

$$s_{r-1} + 1, s_{r-1} + 2, \cdots, s_r.$$

Hence we have $0 \leqq n_{r-1} - n_r \leqq n_{r-1}$. The first inequality shows that the terms of (1) do not increase. This second inequality implies $n_r \geqq 0$. We have just seen that if a term of (1) is zero all further terms vanish. Hence to prove the first part of the theorem it is sufficient to show that the assumption that all terms of (1) are positive leads to a contradiction. This assumption implies that s_r increases indefinitely with r. But C is an infinite class. Hence for r sufficiently large, $n_r = n - \theta(s_r) < 0$ contrary to fact. To complete the proof set $r=k$ and $n_r=0$ in (2) and (3). Then (3) becomes $n_{k-1} = \theta(s_{k-1} + n_{k-1}) - \theta(s_{k-1})$. This tells us that all the numbers

$$s_{k-1} + 1, s_{k-1} + 2, \cdots, s_k$$

belong to C. In particular s_k belongs to C. But by (2), $\theta(s_k) = n$. Hence s_k is the nth member of C.

From the equations defining the terms of (1) one may eliminate the first terms and obtain in this way an expression for n_r which involves only n and θ. In general this expression is very complicated. In particular cases it may be simplified and we obtain from the algorithm a certain identity. One example will suffice to illustrate the method.

Let the set C consist of all positive even integers. Then $\theta(x) = [x/2]$. In this case it is not difficult to show that

$$n_r = \left[\frac{n + 2^{r-1}}{2^r}\right].$$

Hence we have at once the identity

$$n = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \cdots.$$

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NOTE ON TRIPLE SYSTEMS*

BY R. D. CARMICHAEL

A Steiner triple system is said to have the property E if (and only if) the presence of the triples (abu), (bcv), (ucx) implies the presence of the triple (avx) in the system. For such a triple system Th. Skolem† has proved in an elementary way that there exists such a system when and only when the number n of elements involved is of the form $2^{k+1}-1$, where k is a positive integer and that for every n of such form there exists but one triple system with the property E. H. Hasse,† by means of Abelian groups, has given a simple proof of this theorem of Skolem.

Since a Steiner triple system formed from the n elements $x_1, x_2, \cdots, x_n$ has the property that each pair x_ix_j of elements in the set belongs to one and just one triple of the set forming the triple system and since this system is now to be restricted so as to have the property E it follows that the elements $x_1, \cdots, x_n$ and the triples of the system may be taken to be the points and the lines respectively of a finite geometry in the sense of Veblen and Bussey,‡ as one may readily see by comparing the properties of the system with the postulates of the geometry. The property E of the system is equivalent to the transversal property in the geometry.

Since the geometry contains just three points on a line it follows from the general theorem of Veblen and Bussey concerning the existence of finite geometries that this geometry is precisely their $PG(k, 2)$ consisting of $2^{k+1}-1$ points arranged in lines of three points each. These triples of points, constituting the lines of the geometry, form the required triple system with the property E and the system is unique. Thus it follows that the theorem of Skolem is essentially a special case of the existence theorem for the finite geometries.

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* Presented to the Society, September 2, 1932.

† Th. Skolem, Norsk Matematisk Tidsskrift, vol. 13 (1931), pp. 41–51; H. Hasse, ibid., vol. 13 (1931), pp. 105–107.

‡ Veblen and Bussey, Transactions of this Society, vol. 7 (1906), pp. 241–259.

The connection of these triple systems with Abelian groups of order 2^k and type $(1, 1, \cdots, 1)$, brought out by Hasse (loc. cit.), is a special case of a general result which I have established.*

The largest permutation group on $x_1, x_2, \cdots, x_n$ each element of which transforms the triple system into itself is said to be a *triple* group and to be the group *belonging* to the triple system. From the theory of the finite geometries $PG(k, 2)$ it follows at once that the triple group G belonging to the triple system of Skolem is the collineation group in $PG(k, 2)$. When $k>1$ the group G is doubly transitive. (It is obvious that no triple group of degree greater than 3 can be triply transitive.) Moreover, the group G contains a cyclic permutation on all its elements, such a permutation arising readily from a primitive mark of the Galois field $GF[2^k]$. By means of such a cyclic permutation in G any particular triple in the system is transformed successively into all the triples of the system.

The euclidean finite geometry $EG(k, 3)$, in the notation of Veblen and Bussey, also affords an example of a triple system, the lines in the geometry constituting the triples in the system. The elements from which this triple system is formed are 3^k in number, where k is a positive integer. The triple group belonging to the system is the collineation group in the geometry; it is doubly transitive when $k>1$.

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* R. D. Carmichael, American Journal of Mathematics, vol. 52 (1930), pp. 745–788; see pp. 746 ff.

QUADRATIC PARTITIONS: PAPER IV

BY E. T. BELL

1. *Identity of Degree* 5. For the preceding note, see this Bulletin, vol. 38, p. 569. The *degree* of a ϑ, ϕ identity is the degree of the identity in functions ϑ, ϕ. In II we discussed an identity of degree 4. Here we consider an identity of degree 5 whose equivalent in parity functions refers to $F(w, z, u, v)$ as in II. The identity is one of many of degree 5 by Gage.

Denote by $\Psi(w, z, u, v)$ the function

$$\phi_{111}\left(\frac{w+z}{2},\ u\right)\vartheta_3^2\left(\frac{v-z}{2}\right)\vartheta_3^2\left(\frac{w-u}{2}\right).$$

Then

$$\Psi(w, z, u, v) + \Psi(w, -z, -u, -v) - \Psi(w, -z, v, u) - \Psi(w, z, -v, -u) \equiv 0$$

is an identity in w, z, u, v. The required expansions are

$$\vartheta_3(x) = \sum q^{r^2} \cos 2rx, \text{ and}$$

$$\phi_{111}(x, y) = \operatorname{ctn} x + \operatorname{ctn} y + 4\sum q^{2n}\Big[\sum \sin 2(dx + \delta y)\Big].$$

For the notation in the above and in what follows, refer to I.

2. *Equivalent of Ψ-Identity.* To apply the formulas in I, §7, to the reduction of the ctn terms, make the substitution $(w, z) \to (x+y, x-y)$, and in the result apply

$$(x, y) \to ((w+z)/2,\ (w-z)/2).$$

Proceeding as in II, we find the following. The partitions are

$$n = 2d\delta + \nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

Write

$$\lambda_1 \equiv \nu_1 + \nu_2,\ \lambda_2 \equiv \nu_3 + \nu_4,\ \alpha_1 \equiv a_1 + a_2,\ \alpha_2 \equiv a_3 + a_4;$$

$$\sigma_1 \equiv \alpha_1 + \alpha_2,\ \sigma_2 \equiv \alpha_1 - \alpha_2,\ \sigma_{1,r} \equiv \sigma_{2,r} + 2\sigma_2,$$

$$\sigma_{2,r} \equiv 2r - 1 + e(n) - \sigma_2;\ S \equiv \left[\tfrac{1}{2}(|\sigma_1| - 1)\right],\ A \equiv \left[\tfrac{1}{2}(|\alpha_2| - 1)\right].$$

Then the identity gives

$$
\begin{aligned}
&4\sum F(d+\lambda_2, d-\lambda_1, 2\delta-\lambda_2, \lambda_1)\\
&= \sum \operatorname{sgn}\sigma_1\big[e(n)F(\sigma_2/2, \sigma_2/2, \alpha_1, \alpha_2) - F(\alpha_2, \alpha_1, \alpha_1, \alpha_2)\\
&\quad - 2\sum_{r=1}^{S} F(\sigma_{2,r}/2, \sigma_{1,r}/2, \alpha_1, \alpha_2)\big]\\
&\quad + \sum \operatorname{sgn}\alpha_2\big[e(\alpha_2)F(\alpha_2, \alpha_1, \alpha_1, 0) + F(\alpha_2, \alpha_1, \alpha_1, \alpha_2)\\
&\quad + 2\sum_{r=1}^{A} F(\alpha_2, \alpha_1, \alpha_1, 2r-1+e(\alpha_2))\big],
\end{aligned}
$$

with F as in II.

3. *A Summation Formula.* Let $f(x_1, \cdots, x_t)$ be entire in $(x_1, \cdots, x_t)$. Then

$$
\sum_{r=1}^{n} f(a_1 r + b_1, \cdots, a_t r + b_t) = f(a_1\theta(n) + b_1, \cdots, a_t\theta(n) + b_t),
$$

where $\theta(n)$ is the umbra of $\theta_j(n)$ $(j=0, 1, \cdots)$ and $\theta_j(n) \equiv \sum_{r=1}^{n} r^j$. The generators of $\theta(n)$, B are

$$
e^{\theta(n)z} \equiv \frac{e^z(e^{nz}-1)}{e^z-1}, \quad e^{Bz} \equiv \frac{x}{e^z-1}.
$$

Hence

$$
xe^{\theta(n)x} = e^{(n+1+B)x} - e^{(1+B)x}
$$

and therefore, with $\theta_s(n) \equiv 0$, $s<0$,

$$
r\theta_{r-1}(n) = (n+1+B)^r - (1+B)^r, \quad (r=0, 1, \cdots).
$$

4. *Application of §3 to §2.* Denote by $F_2(w, z, u, v)$ the function $F(w, z, u, v)$ with the restriction of entirety in (w, z, u, v). Then

$$
\begin{aligned}
&4\sum F_2(d+\lambda_2, d-\lambda_1, 2\delta-\lambda_2, \lambda_1)\\
&= \sum \operatorname{sgn}\sigma_1\big[e(n)F_2(\sigma_2/2, \sigma_2/2, \alpha_1, \alpha_2) - F_2(\alpha_2, \alpha_1, \alpha_1, \alpha_2)\\
&\quad - 2F_2(\theta(S)+(e(n)-1-\sigma_2)/2, \theta(S)+(e(n)-1+\sigma_2)/2, \alpha_1, \alpha_2)\big]\\
&\quad + \sum \operatorname{sgn}\alpha_2\big[e(\alpha_2)F_2(\alpha_2, \alpha_1, \alpha_1, 0) + F_2(\alpha_2, \alpha_1, \alpha_1, \alpha_2)\\
&\quad + 2F_2(\alpha_2, \alpha_1, \alpha_1, 2\theta(A) + e(\alpha_2) - 1)\big].
\end{aligned}
$$

5. *Contractions.* We contract $F(w, z, u, v)$ with respect to z, as in II, §6. The new partitions are

$$
n = \sum_{i=1}^{4} r_i^2 - 2\delta^2 = 2a_1^2 + a_2^2 + a_3^2 = \sum_{i=1}^{4} b_i^2,
$$

restricted as next stated. Write

$$\lambda_1 \equiv \nu_1 + \nu_2,\ \lambda_2 \equiv \nu_3 + \nu_4,\ \alpha \equiv a_2 + a_3,\ A \equiv [\tfrac{1}{2}(|\alpha| - 1)],$$
$$\beta_1 \equiv b_1 + b_2,\ \beta_2 \equiv b_3 + b_4,\ \sigma_1 \equiv \beta_1 + \beta_2,\ \sigma_2 \equiv \beta_1 - \beta_2.$$

Then the restrictions are

$$\lambda_1 > 2\delta,\ e(n-1) > \sigma_2 \geqq 2 - |\sigma_1|,\ \sigma_2 \equiv e(n-1) \bmod 2.$$

With G, G_1 as in II, we have

$$2\sum G(\lambda_1 + \lambda_2 - 2\delta,\ 2\delta - \lambda_2,\ \lambda_1 - 2\delta)$$
$$= \sum \operatorname{sgn}\alpha\left[\sum_{r=1}^{A} G(\alpha, 0, 2r - e(n-1))\right]$$
$$+ \sum \operatorname{sgn}\sigma_1 G(\sigma_2, \beta_1, \beta_2);$$
$$2\sum G_1(\lambda_1 + \lambda_2 - 2\delta,\ 2\delta - \lambda_2,\ \lambda_1 - 2\delta)$$
$$= \sum \operatorname{sgn}\alpha G_1(\alpha, 0, 2\theta(A - e(n-1))$$
$$+ \sum \operatorname{sgn}\sigma_1 G_1(\sigma_2, \beta_1, \beta_2).$$

In the reductions we have used

$$1 - e(\nu) = e(\nu - 1),$$
$$2[\tfrac{1}{2}(|\nu| - 1)] = |\nu| - 1 - e(\nu),\ \sigma_1 \equiv n \bmod 2.$$

Alternative forms are obtained on noticing that

$$\operatorname{sgn} wF(w, z, u, v) = F(|w|, z, u, v).$$

CALIFORNIA INSTITUTE OF TECHNOLOGY

HEPTAGONAL SYSTEMS OF EIGHT LINES IN A PLANE*

BY LOUISE D. CUMMINGS

1. *Introduction.* This paper concerns the determination of the non-equivalent systems of eight real lines in a plane and the investigation is limited to that subdivision of the problem where in a system of eight lines, some seven form a convex heptagon. The eight marks of a system are used to show the non-equivalence of the *fifteen* heptagonal systems derived in this paper. The notation and method employed are explained fully in the two papers† cited below.

2. *The Cases Considered.* Two subdivisions of the problem are considered:

CASE I. *The secant crosses the heptagon.*

CASE II. *The secant lies outside the heptagon.*

CASE I. The secant crosses two primary, two secondary, two tertiary, and one quaternary segment. The two primary segments may be (i) adjacent, (ii) alternate, (iii) opposite sides of the heptagon.

Case I furnishes ten non-equivalent systems, (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), tabulated below in Table I.

CASE II. The secant crosses the segments as follows: (i) seven quaternary, (ii) five quaternary and two tertiary, (iii) three non-consecutive quaternary and four tertiary, (iv) three consecutive quaternary and four tertiary, (v) one quaternary and six tertiary, and furnishes five non-equivalent systems (11), (12), (13), (14), (15), respectively tabulated in Table I. If the secant crosses (vi) two secondary, four tertiary and one quaternary; (vii) two secondary, two tertiary and three quaternary; (viii) four secondary, two tertiary and one quaternary; three systems are obtained equivalent respectively to systems (3), (4), (2). These results were to be expected since now a second heptagon including the secant line l_8 exists in the system and this may be used as basic heptagon (see Fig. 1).

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* Presented to the Society, March 26, 1932.

† H. S. White, *The plane figure of seven real lines*, this Bulletin, vol. 38 (1932), pp. 59–65; Louise D. Cummings, *Hexagonal systems of seven lines in a plane*, this Bulletin, vol. 38 (1932), pp. 105–110.

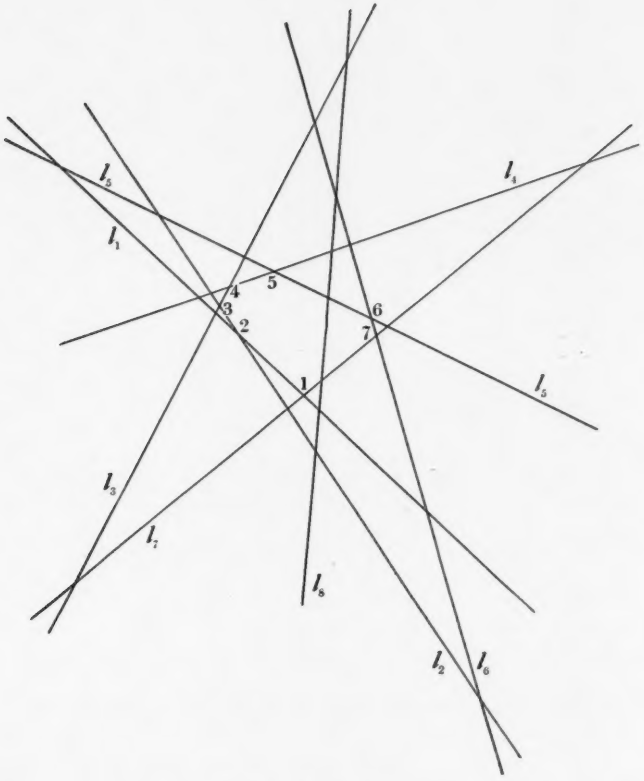

Figure 1. System (3)

TABLE I

System	Characteristics	Polygons					
		S_8	S_7	S_6	S_5	S_4	S_3
(1)	$(4111)^8$	1	0	0	0	20	8
(11)	$(70)+(43)^7$	0	1	0	7	7	14
(15)	$(61)^5+(421)^3+(4111)$	0	1	0	5	11	12
(12)	$(52)+(43)^3+(421)^4$	0	1	0	5	11	12
(14)	$(61)^2+(43)^2+(421)^4$	0	1	0	5	11	12
(13)	$(421)^6+(4111)+(2221)$	0	1	0	3	15	10
(3)	$(511)+(421)^2+(4111)^2+(3211)^3$	0	1	0	2	17	9
(4)	$(421)^2+(4111)^2+(322)+(3211)^3$	0	1	0	2	17	9
(2)	$(4111)^4+(3211)^6$	0	1	0	1	19	8
(6)	$(70)+(511)^2+(43)+(331)^2+(3211)^2$	0	0	1	5	12	11
(7)	$(52)+(511)+(421)+(4111)+(331)^2+(3211)^2$	0	0	1	4	14	10
(5)	$(511)^2+(43)+(421)+(331)+(3211)^3$	0	0	1	4	14	10
(9)	$(511)+(421)+(4111)+(322)+(3211)^4$	0	0	1	3	16	9
(10)	$(511)^2+(4111)+(331)+(3211)^4$	0	0	1	3	16	9
(8)	$(4111)^3+(3211)^4+(2221)$	0	0	1	2	18	8

3. *The Figure.* Examination of the basic diagram shows that a secant which intersects four secondary segments must also intersect two tertiary segments. Therefore all types of paths for the eighth line have been investigated in the preceding eighteen cases.

The numbers listed for the polygonal divisions of the plane for these fifteen heptagonal systems show pairs of systems, for example (3) and (4), with the same polygonal numbers but with different characteristics. Therefore equality of polygonal numbers is a necessary but not a sufficient condition for the equivalence of two systems.

4. *Conclusion.* This method of analysis of straight-line nets by the contiguous segments as herein extended to all the lines in the system regardless of their relation to a heptagon or other basal polygon is applicable to any number n of straight lines and is even not restricted to the case that only two lines shall pass through a point. This method simplifies the discovery of the substitution which transforms one of two equivalent systems into the other.

VASSAR COLLEGE

A NOTE ON NORMAL DIVISION ALGEBRAS OF ORDER SIXTEEN*

BY A. A. ALBERT

1. *Introduction.* I have proved† that every normal division algebra of order sixteen over any non-modular field F contains a quartic field with G_4 group. This important result gave a *determination of all normal division algebras of order sixteen.* I have recently proved‡ the existence of *non-cyclic* normal division algebras, so that the result mentioned above is actually the best possible result. However, my proof of 1929 is long and complicated and the above result there obtained is of sufficient importance to make a better proof desirable. It is the purpose of this note to provide such a proof.

2. *Results Presupposed.*§ We shall require certain well known results on normal division algebras D of order sixteen over F. Algebra D has rank four, so that every sub-field of D is either a quartic field, a quadratic field, or F itself. We also have the following theorems.

THEOREM 1. *Every root in D of the minimum equation of a quantity x of D is a transform yxy^{-1} of x by y in D.*

THEOREM 2. *If $\phi(\omega)=0$ is the minimum equation of x in D, then there exist quantities $x_i=x_1, x_2, \cdots, x_r$ in D such that*

$$\phi(\omega) \equiv (\omega - x_r) \cdots (\omega - x_2)(\omega - x_1).$$

It is obvious that no two adjacent transforms x_{i+1}, x_i in the above irreducible equation can be equal.

From the well known Wedderburn theorem on linear associative algebras with normal simple sub-algebras we have immediately the following result.

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* Presented to the Society, August 31, 1932.

† Transactions of this Society, vol. 31 (1929), pp. 253–260.

‡ See this Bulletin of June 1932 and the Transactions of this Society of January 1933 for proof.

§ See my paper, Annals of Mathematics, vol. 30 (1929), pp. 322–338, for a proof of Theorem 1 and a reference to proofs of Theorem 2 and the Wedderburn result implying Theorem 3.

THEOREM 3. *Let D contain a generalized quaternion algebra Q. Then $D = Q \times R$, where R is also a generalized quaternion algebra.*

3. *On Quartic Fields.* Let X be a quartic field over F. If $X = F(x)$, so that x generates X, then x is called a *primitive* quantity of X, otherwise *imprimitive*. Similarly X is *primitive* if it contains no imprimitive quantities not in F and otherwise is *imprimitive*. The *group* of X is *primitive* or *imprimitive* according as X is primitive or imprimitive.* In a quartic field X the only possible sub-fields except F are those of order two. Hence X is imprimitive if and only if it has a quadratic sub-field.

If x is any primitive quantity of X and the minimum equation of x is $\phi(\omega) = 0$, then X is known† to be imprimitive if X contains an $x_2 \neq x$ and yet satisfying $\phi(\omega) = 0$.

Of particular importance for us are quartic fields X with group

$$(1) \quad G_4 = (I, s_1, s_2, s_3),\ s_1^2 = s_2^2 = s_3^2 = I,\ s_3 = s_1 s_2 = s_2 s_1.$$

Obviously the above group is the direct product of two cyclic groups of order two. Correspondingly, X is the direct product of two quadratic fields. Conversely, every direct product of two non-equivalent quadratic fields is a quartic field with group G_4. We shall prove that every normal division algebra D of order sixteen over D contains two quadratic sub-fields $F(u)$, $F(v)$ such that $uv = vu$, while v is not in $F(u)$. It follows that the sub-field $F(u, v)$ of the division algebra D is the direct product of $F(u)$ and $F(v)$ and has group G_4.

4. *The Existence of a Quadratic Sub-Field of D.* We shall first prove the following lemma.

LEMMA. *Let x in D have*

$$(2) \quad \phi(\omega) \equiv \omega^4 + \alpha\omega^3 + \beta\omega^2 + \gamma\omega + \delta = 0, \quad (\alpha, \cdots, \delta \text{ in } F),$$

as its minimum equation and let Y be a quartic field such that

$$(3) \quad \phi(\omega) \equiv B \cdot A$$

where

$$(4) \quad A \equiv \omega^2 - t_1\omega + s_1,\ B \equiv \omega^2 - t_2\omega + s_2$$

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* See Weber's *Algebra*, vol. 1, p. 505 and p. 525 for the preceding results.

† See Netto's *The Theory of Substitutions*, p. 198.

have coefficients t_1, t_2, s_1, s_2 in Y. Then one of $X=F(x)$ and Y is imprimitive.

For let W be an extension of Y in which A and B are reducible. Then $A\equiv(\omega-\xi_2)(\omega-\xi_1), B\equiv(\omega-\xi_4)(\omega-\xi_3)$, where $\xi_1, \cdots, \xi_4$ are in W so that the fields $X=F(x)$, $F(\xi_i)$ are equivalent quartic fields. If $s_1=\lambda$ in F, then $\xi_2\xi_1=\lambda$, $\xi_2=\lambda\xi_1^{-1}\neq\xi_1$ is in $F(\xi_1)$. Hence $F(\xi_1)$ and $F(x)$ are imprimitive by the argument of §3.

If $s_2+s_1=\lambda$ in F, then, since $s_2s_1=\delta$ in F, s_1 is a root of $\omega^2-\lambda\omega+\delta=0$ and, since s_1 is not in F, Y containing s_1 is imprimitive. Hence let $y=s_1+s_2$ be not in F.

But $y=\xi_4\xi_3+\xi_2\xi_1$ is well known,* by elementary theory of quartic equations, to be a root of the resolvent cubic of $\phi(\omega)=0$. Hence $F(y)$ cannot have order four. Hence Y is imprimitive. Our proof of the lemma is complete.

Let now x be in D and $X=F(x)$ be a *primitive* quartic field. By Theorem 2

$$\phi(\omega)\equiv(\omega-x_4)(\omega-x_3)(\omega-x_2)(\omega-x_1),\ x_1=x, \tag{5}$$

where x_2, x_3, x_4 are in D. Also

$$\begin{gathered}\phi(\omega)\equiv B\cdot A,\quad B\equiv(\omega-x_4)(\omega-x_3)\equiv\omega^2-t_2\omega+s_2,\\ A\equiv(\omega-x_2)(\omega-x_1)\equiv\omega^2-t_1\omega+s_1.\end{gathered} \tag{6}$$

Using (6) and (2) and comparing coefficients of ω^3, ω^2, $\omega^0=1$, respectively, we obtain

$$t_2+t_1=-\alpha,\ s_2+s_1+t_2t_1=\beta,\ s_2s_1=\delta. \tag{7}$$

By substituting for s_2 and t_2 from the first and third parts of (7) in its second part we obtain

$$\delta s_1^{-1}+s_1=\beta+t_1\alpha+t_1^2. \tag{8}$$

Suppose that both $F(s_1)$ and $F(t_1)$ are *primitive*. Then any quantity of each field generates the field and the two fields are equal if and only if they have a quantity not in F in common. But $t_1^2+\alpha t_1+\beta$ is not in F and is in $F(s_1)$ by (8). Hence $F(t_1)=F(s_1)=Y$ contains all of s_1, s_2, t_1, t_2 by (7). But X is a primitive field by hypothesis. Hence, by the above lemma, Y is imprimitive, a contradiction.

* See L. E. Dickson, *First Course in the Theory of Equations*, p. 51. Thus the above essential point of our proof is a very elementary result.

Hence at least one of $F(s_1)$, $F(t_1)$ is imprimitive and contains a quadratic sub-field, or one of these fields is a quadratic field, or one of them coincides with F. But if either $t_1=x_2+x_1$ or $s_1=x_2x_1$ is in F, then X contains a root $x_2\neq x_1$ of the minimum equation of x_1 and is imprimitive, a contradiction.

Hence if x is in D but not in F, then either $F(x)$ is a quadratic field, or $F(x)$ is imprimitive and contains a quadratic field, or $F(x)$ is primitive but either $F(s_1)$ or $F(t_1)$ contains a quadratic field. We have thus proved the following theorem.

THEOREM 4. *Every normal division algebra of order sixteen over F contains a quadratic sub-field.*

The above proof is certainly a great simplification of my earlier proof of the same theorem requiring about four printed pages. We shall also obtain a simpler proof of the final result, a consequence of Theorem 4.

5. *The Desired Determination.* Let D be a normal division algebra of order sixteen over F. By Theorem 4, D contains a quadratic sub-field $U=F(u)$, $u^2=\rho$ in F. But $(-u)^2=\rho$ so that, by Theorem 1, $-u=yuy^{-1}$, where y is in D. From $yu=-uy$ it follows that $y^2u=uy^2$. If $y^2=\lambda$ in F, algebra D contains the generalized quaternion algebra $Q=(1, u, y, uy)$ over F and, by Theorem 3, $D=Q\times R$, where $R=(1, v, z, vz)$, $zv=-vz$, $v^2=\sigma$ in F. In this case D contains the quartic field $F(u, v)$ with G_4 group as desired. Let then $y^2=v$ not in F. If v is a primitive quantity of $Y=F(y)$, then Y is a polynomial in y^2. But y^2 is commutative with u while y is not commutative with u. Hence v is imprimitive, $vu=uv$, $F(v)$ is a quadratic field. Also u is obviously not in $F(v)<F(y)$ so that $F(u, v)$ is a quartic field with group G_4. The following theorem is thus proved.

THEOREM 5. *Every normal division algebra of order sixteen over any non-modular field F contains a quartic sub-field with group G_4.*

THE UNIVERSITY OF CHICAGO

ON UNIT-ZERO BOOLEAN REPRESENTATIONS OF OPERATIONS AND RELATIONS*

BY B. A. BERNSTEIN

1. *Introduction.* Consider an algebra $(K, +, \times)$, such as ordinary real algebra, in which there are two elements "0" and "1" having the properties that, for any element a,

$$a + 0 = 0 + a = a, \; a1 = 1a = a. \tag{1}$$

Let

$$(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)$$

denote a *unit-zero function with respect to the sequence of m elements*, $a_1, \cdots, a_m$ of K, that is, a function $f(x_1, x_2, \cdots, x_m)$ of m elements $x_1, x_2, \cdots, x_m$ such that $f=1$ or 0, according as the equalities, $x_i=a_i$, $(i=1, 2, \cdots, m)$, all hold or do not all hold. Accordingly, $(x; a)$ will denote a *unit-zero function with respect to* a, that is, a function $f(x)$ such that $f(x)=1$ or 0, according as $x=a$ or $x\neq a$. Then the following propositions (2)–(4) evidently hold:

$$(2)\ (x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) = (x_1; a_1)(x_2; a_2) \cdots (x_m; a_m);$$

$$(3)\qquad a(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) = a \text{ or } 0,$$

according as $x_i=a_i$, $(i=1, 2, \cdots, m)$, all hold or do not all hold;

$$(4)\ a(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) + b(x_1, x_2, \cdots, x_m; b_1, b_2, \cdots, b_m) = a, \text{ or } b, \text{ or } 0,$$

according as $x_i=a_i$ all hold, or $x_i=b_i$ all hold, or neither $x_i=a_i$ all hold nor $x_i=b_i$ all hold, $(i=1, 2, \cdots, m; a_i\neq b_i$ for some $i)$. In a previous paper† propositions (1)–(4) were made the basis of a method of obtaining arithmetic representations of arbitrary operations and relations in a finite class of elements. Since

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* Presented to the Society, April 11, 1931.

† B. A. Bernstein and N. Debely, *A practical method for the modular representation of finite operations and relations*, this Bulletin, vol. 38 (1932), pp. 110–114.

propositions (1)–(4) also hold when the symbols belong to Boolean algebra, the question naturally arises: To what extent can unit-zero functions be used analogously to obtain *Boolean* representations of arbitrary operations and relations? The object of the present paper is to answer this question.

2. *Determination of Boolean Unit-Zero Algebras.* The possibility of representing arbitrary operations and relations by unit-zero functions of an algebra hinges on the existence in this algebra of a unit-zero function for every sequence of m of its elements. Let us call an algebra which has a unit-zero function for every sequence of m of its elements a *unit-zero algebra.* I proceed first to determine all Boolean unit-zero algebras.

This determination is made easy by noting at the outset that a unit-zero Boolean function must satisfy proposition (2) above and also that it must be single-valued. We therefore need to look only for Boolean unit-zero functions $f(x)$ of a *single* variable x of the form*

$$(5) \qquad (x; a) = (1; a)x + (0; a)x'.$$

From (5) we see, by putting $a = 0, 1$, that in a Boolean algebra of *two* elements, x is the unit-zero function of x with respect to 1, and x' is the unit-zero function of x with respect to 0; in symbols,

$$(6) \qquad (x; 1) = x, \quad (x; 0) = x'.$$

We have, then, that *a two-element Boolean algebra is a unit-zero algebra, the unit-zero functions of one variable x being given by* (6).

By (2) and (6), *all* the unit-zero functions of a two-element Boolean algebra can be readily written down. Thus, the unit-zero functions of two variables x, y are given by

$$(7) \qquad \begin{aligned} (x, y; 1, 1) &= xy, \quad (x, y; 1, 0) = xy', \\ (x, y; 0, 1) &= x'y, \quad (x, y; 0, 0) = x'y'. \end{aligned}$$

In general, *the unit-zero functions of m variables are the* 2^m *constituents in the normal development of* 1 *with respect to the m variables.*

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* The usual Boolean notations are employed: $a+b$, ab, a', 0, 1 are respectively the *sum* of a and b, the *product* of a and b, the *negative* of a, the *zero* element, the *whole.*

Let us now consider a Boolean algebra A of *more* than two elements. A must have an element $e \neq 0, 1$. Suppose, first, that A has a unit-zero function $f(x)$, of form (5), with respect to e. Then

$$\text{(i)} \qquad f(e) = 1, f(0) = 0, f(1) = 0, \qquad (e \neq 0, 1).$$

But (i) is inconsistent with (5). Hence, *our algebra A has no unit-zero function with respect to a sequence containing the element e.*

Suppose, next, that the algebra A has a unit-zero function $f(x)$, of form (5) with respect to 0. Then

$$\text{(ii)} \qquad f(0) = 1, f(1) = 0, f(e) = 0, \qquad (e \neq 0, 1).$$

Hence, by (5),

$$\text{(iii)} \qquad f(x) = x', f(e) = 0, \qquad (e \neq 0, 1).$$

But equations (iii) are inconsistent. Hence, *our algebra A has no unit-zero function with respect to a sequence containing the element* 0.

Similarly, *our algebra A has no unit-zero function with respect to a sequence containing the element* 1. Hence, *a Boolean algebra of more than two elements has no unit-zero functions at all.*

Our main result is, then, the following theorem.

THEOREM A. *The only Boolean unit-zero algebra is a two-element Boolean algebra.*

3. *Dual Considerations.* By the Principle of Duality in Boolean algebras each of the foregoing propositions about unit-zero Boolean functions has a dual proposition corresponding to it. To state these duals, let me use the notion of *zero-unit function* (to be distinguished from *unit-zero function*). By a *zero-unit function of* $x_1, x_2, \cdots, x_m$ *with respect to the sequence* $a_1, a_2, \cdots, a_m$, symbolized by

$$[x_1, x_2; \cdots, x_m; a_1, a_2, \cdots, a_m],$$

let us mean a function $f(x_1, x_2, \cdots, x_m)$ such that $f = 0$ or 1, according as $x_i = a_i$, $(i = 1, 2, \cdots, m)$, all hold or do not all hold. The duals of (2), (3), and (4) are, then, respectively (2′), (3′), and (4′) following:

$$(2') \quad [x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m] = [x_1; a_1] + [x_2; a_2] + \cdots + [x_m; a_m];$$

$$(3') \quad a + [x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m] = a \text{ or } 1,$$

according as $x_i = a_i$, $(i = 1, 2, \cdots, m)$, all hold or do not all hold;

$$(4') \quad \{a + [x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m]\} \cdot \{b + [x_1, x_2, \cdots, x_m; b_1, b_2, \cdots, b_m]\} = a, \text{ or } b, \text{ or } 1,$$

according as $x_i = a_i$ all hold, or $x_i = b_i$ all hold, or neither $x_i = a_i$ all hold nor $x_i = b_i$ all hold, $(i = 1, 2, \cdots, m;\ a_i \neq b_i$ for some $i)$.

The dual of Theorem A is

THEOREM A′. *The only zero-unit Boolean algebra is a two-element Boolean algebra.*

For a *two-element* Boolean algebra we have, further:

$$(6') \quad [x; 0] = x, \qquad [x; 1] = x';$$

$$(7') \quad [x, y; 0, 0] = x + y, \quad [x, y; 0, 1] = x + y',$$
$$[x, y; 1, 0] = x' + y, \quad [x, y; 1, 1] = x' + y'.$$

In general, *the zero-unit functions of m variables are the* 2^m *factor-constituents in the dual normal development of* 0 *with respect to the m variables.*

Propositions (2′)–(7′) will be used below in the representation of operations that do not satisfy the condition of closure.

4. *Representations.* It is now clear to what extent we can apply unit-zero Boolean functions in the representation of arbitrary operations and relations. From Theorem A, we have

THEOREM B. *A unit-zero Boolean representation of arbitrary operations and relations is possible when and only when the class consists of two elements.*

For a two-element class K, the theory of Boolean representation follows from propositions (2)–(7) and their duals. If we denote the two K-elements by the Boolean symbols 0, 1, the representations of all operations O and relations R in K are covered by the cases 1–3 following.

CASE 1. *O an m-ary operation satisfying the condition of closure.* There is a K-element, 0 or 1, for every sequence $e_1, e_2, \cdots, e_m$ taken from K. Let the sequences to which 1 corresponds be

(i) $\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1m}; \alpha_{21}, \alpha_{22}, \cdots, \alpha_{2m}; \cdots; \alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{km}.$

The representation of O is the Boolean function

$$(8) \qquad \sum_{i=1}^{k}(x_1, x_2, \cdots, x_m; \alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im}).$$

CASE 2. *O an m-ary operation not satisfying the closure condition.* There are sequences in K to which no K-elements correspond. Let these sequences be

(ii) $\beta_{11}, \beta_{12}, \cdots, \beta_{1m}; \beta_{21}, \beta_{22}, \cdots, \beta_{2m}; \cdots; \beta_{k1}, \beta_{k2}, \cdots, \beta_{km}.$

Consider the operation O' obtained from O by assigning a K-element, 0 for convenience, to each of the sequences (ii). Let $\phi(x_1, x_2, \cdots, x_m)$, obtained as in Case 1, be the representation of O'. Then the representation of O is the function

$$(9) \quad \phi(x_1, x_2, \cdots, x_m) + \sum_{i=1}^{k} 0/[x_1, x_2, \cdots, x_m; \beta_{i1}, \beta_{i2}, \cdots, \beta_{im}],$$

where a/b means the unique K-element q satisfying the condition $bq=a$.*

CASE 3. *R an m-adic relation.* Let the sequences which do *not* satisfy R be

(iii) $\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1m}; \gamma_{21}, \gamma_{22}, \cdots, \gamma_{2m}; \cdots; \gamma_{k1}, \gamma_{k2}, \cdots, \gamma_{km}.$

Then the representation of R is the Boolean equation

$$(10) \qquad \sum_{i=1}^{k}(x_1, x_2, \cdots, x_m; \gamma_{i1}, \gamma_{i2}, \cdots, \gamma_{im}) = 0.\dagger$$

Of course, by the Duality Principle, the theory of representation can be stated primarily in terms of zero-unit functions instead of unit-zero functions.

5. *Illustrations.* The following illustrations, one for each of the above three cases, will make the theory of representation quite clear.

α. Let O be the operation defined by

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* For a two-element Boolean algebra the *quotient* can be defined precisely as in ordinary algebra.

† Instead of 0, we can use 1 in (10), provided (i) are the sequences which *do* satisfy R.

(i)

	0	1
0	1	0
1	0	1

Its representation is

(ii) $(x, y; 0, 0) + (x, y; 1, 1) \equiv x'y' + xy.$

β. Let O be the operation

(iii)

	0	1
0	0	1
1	−	−

where the blanks indicate that there are no K-elements corresponding to the sequences 1, 0; 1, 1.

Consider the operation O' defined by

(iv)

	0	1
0	0	1
1	0	0

By Case 1, the representation of O' is

(v) $x'y.$

Hence, the representation of O is

(vi) $x'y + 0/[x, y; 1, 0] + 0/[x, y; 1, 1]$
$\equiv x'y + 0/(x' + y) + 0/(x' + y').$

γ. Let R be a relation defined by

(vii)

	0	1
0	−	+
1	+	−

where "+" indicates that R holds and "−" indicates that R does not hold. Its representation is the equation

(viii) $(x, y; 0, 0) + (x, y; 1, 1) \equiv x'y' + xy = 0.$*

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* For a complete set of Boolean representations of binary operations and dyadic relations in a two-element class, obtained from considerations other than the above, see my *Complete sets of representations of two-element algebras*, this Bulletin, vol. 30 (1924), pp. 24–30.

THE POISSON INTEGRAL FOR FUNCTIONS WITH POSITIVE REAL PART*

BY W. CAUER

The aim of this paper is to derive a Poisson integral representation valid for all functions $g(\lambda)$ which are regular in the right λ-half-plane, and for which†

$$\Re g(\lambda) \geqq 0$$

in that half-plane, and in particular for such functions $g(\lambda)$ which are real for real values of λ. This latter class of functions as well as their Poisson integral representations play a fundamental role in the theory of alternating current networks.‡ The resulting Poisson integral representation (equation (10)) is a very simple one and closely connected with the theory of Stieltjes' continued fractions. These facts seem to justify the elementary derivation presented here, though the equivalent Poisson integral for the unit circle is well known.

Herglotz§ has proved¶ the theorem: *Every function $f(z)$ regular in the interior of the unit circle with real part not negative (and only such functions) can be represented as*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\alpha} + z}{e^{i\alpha} - z}\, d\mu(\alpha) + k, \tag{1}$$

the integral being taken in the Stieltjes sense, μ being a non-decreasing bounded function, k a pure imaginary constant.

Taking the real part we obtain the Poisson integral for any

* Presented to the Society, August 31, 1932.

† $\Re$ means "real part of."

‡ W. Cauer, Jahresberichte der deutschen Mathematiker Vereinigung, 1929; Mathematische Annalen, vol. 105 (1931); vol. 106 (1932); O. Brune, Journal of Mathematics and Physics, Mass. Inst. of Tech., 1931.

§ G. Herglotz, *Über Potenzreihen mit positivem reellen Teil im Einheitskreis*, Leipziger Berichte, vol. 63 (1911). Herglotz also gives an explicit expression for $\mu(\phi)$ in terms of the coefficients of the power series for $f(z)$.

¶ Another proof follows by theorems of Helly, Wiener Sitzungsberichte, vol. 121 (IIa) (1912), p. 283 and 288. See also Evans, *The Logarithmic Potential*, 1927, p. 46; Bray, Annals of Mathematics, (2), vol. 20 (1919), p. 180, Theorem 3; T. H. Hildebrandt, this Bulletin, vol. 28 (1922), pp. 53–58.

positive harmonic function in the unit circle

$$(2)\quad u(r,\phi) = \Re f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\alpha)+r^2}\,d\mu(\alpha),$$

where $z = re^{i\phi}$.

The following derivation of the Poisson integral for the right λ-half-plane as mentioned in the beginning depends on formula (1). Since any function of bounded variation can be represented as a difference of two non-decreasing bounded functions, in the sequel $\mu(\phi)$ may be supposed to be a non-decreasing function. For simplicity we redefine $\mu(\phi)$ at all interior points of discontinuity (if necessary) in such a way, that $\frac{1}{2}[\mu(\phi+0)+\mu(\phi-0)] = \mu(\phi)$. The substitution

$$(3)\qquad z = \frac{\lambda-1}{\lambda+1}$$

transforms the interior of the unit circle into the right λ-half-plane, real λ corresponding to real z. Similarly

$$e^{i\alpha} = \frac{iy-1}{iy+1},$$

$$y = \operatorname{ctn}\frac{\alpha}{2},$$

where the interval $0<\alpha<2\pi$ corresponds to $\infty > y > -\infty$, gives a corresponding transformation of the variable of integration.

If we write

$$(4)\qquad -\frac{1}{2\pi}\mu(\alpha) = \nu(y),$$

then the transformed Poisson integral becomes

$$f(z) = g(\lambda) = -\int_{\operatorname{ctn}\epsilon/2}^{-\operatorname{ctn}\epsilon/2} \frac{(iy-1)(\lambda+1)+(iy+1)(\lambda-1)}{(iy-1)(\lambda+1)-(iy+1)(\lambda-1)}\,d\nu(y)$$
$$+\frac{1}{2\pi}\int_0^{\epsilon} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\mu(\alpha) + \frac{1}{2\pi}\int_{2\pi-\epsilon}^{2\pi} \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\mu(\alpha)+k.$$

Since $d\alpha = -2dy/(1+y^2)$, $\nu(y)$ is a non-decreasing function.

Considering the possible jump of $\mu(\phi)$ at $\phi=0$ and $\phi=2\pi$, and the relation

$$\frac{1+z}{1-z}=\lambda,$$

we obtain, by passing to the limit as $\epsilon\to 0$, the following result.

THEOREM 1. *Every function $g(\lambda)$, regular in the right λ-half-plane with $\Re g(\lambda)\geqq 0$ there, can be represented as*

$$g(\lambda)=\int_{-\infty}^{+\infty}\frac{iy\lambda-1}{iy-\lambda}\,d\nu(y)+C\lambda+k, \tag{5}$$

where ν is a non-decreasing bounded function, C a not negative and k a pure imaginary constant.

In connection with Stieltjes' moment problem Rolf Nevanlinna* has studied functions $h(x)$ *regular in the upper x-half-plane having there a non-positive imaginary part.* To obtain a Poisson integral formula for such functions (5) must be transformed by

$$i\lambda=x,$$
$$-ig(\lambda)=h(x),$$

so that†

$$h(x)=\int_{-\infty}^{\infty}\frac{1-xy}{x+y}\,d\nu(y)-Cx+l, \tag{6}$$

where ν is a non-decreasing bounded function, C a not negative

* Rolf Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjes'sche Momentenproblem*, Suomalaisen Tiedeakatemian Kustantama, 1922.

† The more direct but not shorter derivation of R. Nevanlinna (loc. cit, p. 44) gives the result

$$h(x)=\int_{-\infty}^{\infty}\frac{d\chi(u)}{x-u},$$

χ non-decreasing, for all functions $h(x)$ allowing the asymptotic development $c_1/x+c_2/x^2+c_3/x^3+\cdots$ for $\epsilon<\arc x<\pi-\epsilon$, where ϵ is an arbitrary small positive number. This supposition requires the existence of $\int_{-\infty}^{\infty}u^n d\chi(u)$ for each positive n. But, according to (6), Nevanlinna's formula holds if we suppose only that $\lim_{r=\infty}\left|re^{i\phi}h(re^{i\phi})\right|<K$ for some positive constant K and some ϕ of the interval $0<\phi<\pi$ and if $\int_a^{\infty}d\chi(u)/u$ exists for $a>0$. Then $\chi(u)=-\int(1+u^2)d\nu(-u)$.

and l a real constant.

Now $g(\lambda)$ of (5) may be further restricted to be real for real values of λ or, what amounts to the same thing, $f(z)$ real for real z and, by the Schwarz reflection principle,

$$f(\bar{z}) = \bar{f}(z),$$

where the dash indicates the conjugate complex value.

This implies

$$u(r, 2\pi - \phi) = u(r, \phi),\ u = \Re f(z),\ z = re^{i\phi}.$$

From the formula*

$$(7)\quad \lim_{r=1} \int_{\phi_1}^{\phi_2} u(r, \alpha)d\alpha = \mu(\phi_2) - \mu(\phi_1),\quad (0 < \phi_1 < \phi_2 < 2\pi),$$

it follows that

$$\mu(\phi_2) - \mu(\phi_1) = \mu(2\pi - \phi_1) - \mu(2\pi - \phi_2),$$

and therefore by (4),

$$(8)\qquad \nu(-y) = -\nu(y) + \text{const.}$$

Then the integral of (5) can be written

$$\int_0^\infty \left(\frac{iy\lambda - 1}{iy - \lambda} + \frac{-iy\lambda - 1}{-iy - \lambda}\right)d\nu(y) = 2\lambda \int_0^\infty \frac{(1 + y^2)d\nu(y)}{\lambda^2 + y^2}.$$

Since this is real for real λ, $g(\lambda)$ becomes real for real λ when and only when $k = 0$. On defining the non-decreasing function $\psi(x)$ by

$$(9)\qquad \psi(x) = 2\int_0^{\sqrt{x}} (1 + y^2)d\nu(y)$$

we get the result:

THEOREM 2. *Any function $g(\lambda)$ which is regular in the right λ-half-plane and such that $\Re g(\lambda) \geqq 0$ there and $g(\lambda)$ is real for real λ, can be represented as*

$$(10)\qquad g(\lambda) = \lambda\left[C + \int_0^\infty \frac{d\psi(x)}{\lambda^2 + x}\right],$$

* See, for instance, G. C. Evans, loc. cit., p. 37, equation (14).

where C is a non-negative constant, ψ a non-decreasing function and the integral is to be taken in the Stieltjes' sense. Conversely if integral (10) *exists, it represents a function of the class specified.*

The converse part of Theorem 2 can be verified immediately. ψ is not necessarily bounded, but it is obvious that the integral (10) exists if and only if

$$\lim_{b=\infty} \int_a^b \frac{d\psi(x)}{x}$$

exists for $a>0$. Since

$$\int_a^b \frac{d\psi(x)}{x} = \frac{1}{\xi}[\psi(b) - \psi(a)], \qquad (a \leqq \xi \leqq b),$$

a necessary condition is that $\psi(x)/x$ be bounded for $x>a>0$. A sufficient condition for the existence of (10) is that $\psi(x)<Ax^{\alpha}$, where A and α are positive constants and $\alpha<1$.

By (7), ψ is connected with the real part of the integral function of g. The relation is

$$\psi(x) = \frac{2}{\pi} \lim_{\epsilon=0} \Re \int_0^{\sqrt{x}} g(\epsilon + iy)dy.*$$

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* The direct proof of Perron in *Die Lehre von den Kettenbrüchen*, 2d edition, 1929, p. 372–374 is applicable to our case, because Perron's restriction that $\int_0^{\infty} d\psi(x)$ exists is superfluous for his proof, if a slight modification is made.

THREE THEOREMS APPLICABLE TO VIBRATION THEORY*

BY B. F. KIMBALL

1. *Introduction.* In discussions of the theory of free vibrations it is sometimes stated without proof that "the fact that the period of vibration is independent of the amplitude shows that the restoring force is linear." The present paper includes a proof of this statement and a brief consideration of the case of a non-linear restoring force.

2. *Hypotheses and Existence of Period Integral.* (A) Let the restoring force $f(x)$ be a continuous function of the distance x from the position of equilibrium on the interval

$$0 \leqq x \leqq D. \tag{1}$$

Let $f'(x)$ exist on this interval. Let $f(0) = 0$ and

$$f(x) > 0,\ f'(x) > 0 \text{ on } 0 < x \leqq D.$$

We suppose also that $f(x)$ is symmetric in $x = 0$; that is, we define the function over the interval $-D \leqq x \leqq 0$ so that

$$f(+x) = -f(-x).$$

(B) Take the derivative

$$f'(0) = b > 0.$$

(C) Let the unilateral derivative $f''(0^+)$ exist.

The differential equation of vibration is

$$\frac{d^2x}{dt^2} = -f(x),$$

and one obtains as an integral

$$t = \int_0^x (C - 2F(x))^{-1/2} dx + \text{const.},$$

where $F(x) = \int_0^x f(x)dx$ and C is a constant of integration. Let a

* Presented to the Society, March 26, 1932.

denote the maximum displacement from the position of equilibrium where a lies on interval (1). This we call the amplitude of the vibration. Then the integral which applies to the motion is

$$t = 2^{-1/2} \int_0^x (F(a) - F(x))^{-1/2} dx.$$

Thus the period of vibration T is given by

$$T = 2 \cdot 2^{1/2} \int_0^a (F(a) - F(x))^{-1/2} dx, \tag{2}$$

if this integral exists.

The existence of the integral can be established as follows. Set

$$Q(x, a) = (F(a) - F(x))/(a - x), \tag{3}$$

and the integral becomes

$$T = 2 \cdot 2^{1/2} \int_0^a (a - x)^{-1/2} (Q(x, a))^{-1/2} dx. \tag{4}$$

Now $Q(x, a)$ is continuous in x except at $x = a$. Furthermore

$$\lim_{x \to a} Q(x, a) = F'(a).$$

We define

$$Q(a, a) = F'(a) \tag{5}$$

and note that

$$Q(0, a) = F(a)/a. \tag{6}$$

By definition $F(x) < F(a)$ for $x < a$. Hence one concludes that $Q(x, a)$ is a positive continuous function on the interval $0 \leqq x \leqq a$ and the existence of integral (4) follows.

3. *Double Inequality Satisfied by Period.* We wish to establish a double inequality bounding T. Apply the first law of the mean for integrals to (4). This gives

$$T = 2 \cdot 2^{1/2} (Q(\bar{x}, a))^{-1/2} \int_0^a (a - x)^{-1/2} dx = 4 \cdot 2^{1/2} (a/Q(\bar{x}, a))^{1/2}.$$

Now it can be shown that $Q'(x, a)$ is positive on the interval

$0 \leqq x < a$. In order to show this note that

$$Q'(x, a) = \frac{Q(x, a) - F'(x)}{a - x}.$$

Apply the law of the mean to $Q(x, a)$ and we have

$$Q'(x, a) = (F'(u) - F'(x))/(a - x), \quad x < u < a.$$

Since $f'(x)$ is positive, $F'(x)$ is an increasing function of x and thus $Q'(x, a)$ is positive on $0 \leqq x \leqq a$. Hence referring to (5) and (6) we have

$$F(a)/a < Q(\bar{x}, a) < F'(a), \tag{7}$$

and T must satisfy the relation

$$4 \cdot 2^{1/2}(a/F'(a))^{1/2} < T < 4 \cdot 2^{1/2}(a^2/F(a))^{1/2}, \tag{8}$$

or, since $F'(x) = f(x)$,

$$4 \cdot 2^{1/2}(a/f(a))^{1/2} < T < 4 \cdot 2^{1/2}(a^2/F(a))^{1/2}. \tag{9}$$

Hence we may state the following theorem.

THEOREM 1. *The period T of a free vibration due to a restoring force $f(x)$ conditioned by hypothesis* (A) *satisfies the double inequality* (9), *where a is the amplitude and $F(a) = \int_0^a f(x)dx$.*

4. *A Necessary Condition that Period be Independent of Amplitude.* First we note that if $f'(0) = 0$, it follows from the inequality established above that $T \to \infty$ as $a \to 0$. On the other hand if $a \neq 0$, T is finite. Hence *if $f'(0) = 0$, T is not independent of a.*

Before going further we shall establish the following lemma for the sake of clarity.

LEMMA 1. *Given function $\psi(x)$, where $\psi(0) = A$, $\psi'(x) = B$, $\psi''(0) = C$, and where $\psi'(x)$ exists for $x > 0$, $\psi''(x)$ does not necessarily exist for $x > 0$; then*

$$\lim_{x \to 0} (\psi(x) - A - Bx)/x^2 = \tfrac{1}{2}C.$$

In order to prove this lemma write

$$p(x) = \psi(x) - A - Bx - \tfrac{1}{2}Cx^2.$$

Then $p(0) = 0$, $p'(0) = 0$ and $p'(x)$ exists for $x > 0$, $p''(0) = 0$. Now

$$\lim_{x\to 0} (\psi(x) - A - Bx)/x^2 = \tfrac{1}{2}C + \lim_{x\to 0} p(x)/x^2.$$

Write $p(x) = x\,p'(\theta)$, where $0 < \theta < x$. Then it is clear that

$$\lim_{x\to 0} p(x)/x^2 = \lim_{x\to 0} (\theta p'(\theta))/(x\theta) = 0,$$

and the lemma is established.

We now undertake to find a necessary condition that T be independent of a under hypotheses (A) and (B) of §2. The quantity b satisfies the relation $b = f'(0) = F''(0)$. Consider the function $\phi(x)$ defined by the equation

$$\phi(x) = 2F(x)/(bx^2), \quad 0 < x \leqq D.$$

From the above lemma

$$\lim_{x\to 0} 2F(x)/(bx^2) = 1.$$

If $\phi(0) = 1$ by definition, $\phi(x)$ is a continuous function of x on the interval $0 \leqq x \leqq D$. Thus $\phi(x)$ takes on its absolute maximum and minimum values on this interval. Let $\phi(x_0)$ be the maximum and $\phi(x_1)$ the minimum. Denote by $T(z)$ the period of vibration due to force $f(x)$ with amplitude z. Comparing $\phi(x)$ and $\phi(x_0)$ we have

$$F(x)/F(x_0) \leqq x^2/x_0^2, \quad 0 \leqq x \leqq x_0. \tag{10}$$

Hence

$$\int_0^{x_0} (1 - F(x)/F(x_0))^{-1/2} dx \leqq \int_0^{x_0} (1 - x^2/x_0^2)^{-1/2} dx. \tag{11}$$

Again, since $\phi(0) = 1$, we judge that $(F(x_0))^{-1/2} \leqq (\frac{1}{2}bx_0^2)^{-1/2}$. Multiplying this inequality by (11) one finds that

$$T(x_0) \leqq 4/(x_0 b^{1/2}) \int_0^{x_0} (1 - x^2/x_0^2)^{-1/2} dx = 2\pi b^{-1/2}. \tag{12}$$

By a similar argument one finds that

$$T(x_1) \geqq 2\pi b^{-1/2}. \tag{13}$$

Now, if $\phi(x_0) > 1$, the inequality sign will apply to (10) for some values of x, and thus the inequality sign will have to be used in (12). Similarly, if $\phi(x_1) < 1$, the inequality sign must be used

in (13). Hence, if $T(z)$ is to be independent of z, we must have $\phi(x) \equiv 1$, which means that $f(x) \equiv bx$. Thus we have the following theorem.

THEOREM 2. *In order that the period of a vibration produced by a force $f(x)$, conditioned by hypotheses* (A) *and* (B), *be independent of the amplitude it is necessary and sufficient that $f(x)$ be a linear function of x of the form bx.*

5. *Limit Approached by Period as Amplitude Approaches Zero.* In this section we employ hypotheses (A), (B) and (C) of §2. We find it convenient to use the function $\phi(x)$ defined above. We write

$$F(x) = \tfrac{1}{2}bx^2\phi(x),$$

$$F(a) = \tfrac{1}{2}ba^2\phi(a),$$

and

$$\phi(x) = \phi(a) - (a - x)\phi'(u_2), \quad x < u_2 < a.$$

Now

$$\phi'(0) = \lim_{x \to 0} (\phi(x) - \phi(0))/x = (2/b) \lim_{x \to 0} (F(x) - \tfrac{1}{2}bx^2)/x^3.$$

Thus using Lemma 1 extended to apply to the third derivative, we find that this limit is $F'''(0^+)/(3b)$ and hence

$$\phi'(0^+) = f''(0)/(3b). \tag{14}$$

Thus $\phi'(x)$ is bounded on the interval

$$0 \leqq x \leqq a. \tag{15}$$

We write

$$F(a) - F(x) = \tfrac{1}{2}b\phi(a)(a^2 - x^2) + \tfrac{1}{2}bx^2(a - x)\phi'(u_2),$$

and note that from the above discussion $\lim_{a \to 0} \phi'(u_2)x^2/(a+x)$ is equal to zero and uniformly convergent for x on (15). Also $\lim_{a \to 0} \phi(a) = 1$. Hence

$$F(a) - F(x) = \tfrac{1}{2}b(a^2 - x^2)(1 + e(x, a)),$$

where $e(x, a)$ converges uniformly to zero with a for x on (15). Accordingly

$$(F(a) - F(x))^{-1/2} = (\tfrac{1}{2}b(a^2 - x^2))^{-1/2}(1 + h(x, a)),$$

where $h(x, a)$ converges uniformly to zero with a for x on (15).

One is thus led to the conclusion that

$$\lim_{a \to 0} 2 \cdot 2^{1/2} \int_0^a (F(a) - F(x))^{-1/2} dx = 2\pi b^{-1/2}.$$

Hence we have the following theorem.

THEOREM 3. *The period of vibration T under restoring force $f(x)$, conditioned by hypotheses* (A), (B), *and* (C), *approaches the limit $2\pi b^{-1/2}$ as the amplitude approaches zero.*

SCHENECTADY, N. Y.

A NOTE ON FERMAT'S LAST THEOREM

D. H. LEHMER*

In 1925 H. S. Vandiver† proved the following theorem.

THEOREM 1. *If*

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied by integers x, y, z, prime to the odd prime p, then the first factor of the class number of the field generated by $e^{2\pi i/p}$ is divisible by p^8.

In the seventh of a series of articles on Fermat's last theorem, T. Morishima‡ has given the following improvement upon Theorem 1.

THEOREM 2. *In Theorem* 1 *we may replace p^8 by p^{12} provided p does not divide* $75571 \cdot 20579903$.

It is the purpose of this note to show that the proviso of Theorem 2 is unnecessary by showing that (1) is not satisfied by the prime factors of $75571 \cdot 20579903$. This is done by applying Wieferich's‖ criterion.

THEOREM 3. *If* (1) *is satisfied by integers x, y, z, prime to p, then* $2^{p-1} \equiv 1 \pmod{p^2}$.

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* National Research Fellow.

† Annals of Mathematics, (2), vol. 26, p. 232.

‡ Proceedings of the Imperial Academy of Japan, vol. 8 (1932), pp. 63–66.

‖ Journal für Mathematik, vol. 136 (1909), p. 203.

In the first place

$$p_1 = 75571 \text{ and } p_2 = 20579903$$

are prime numbers. To show that p_2 is a prime we observe that $2^{(p_2-1)/563} = 2^{36554} \equiv 18351241 \equiv r \pmod{p_2}$ and that $r-1$ and p_2 are relatively prime. By the congruence (2) below we have

$$2^{p_2-1} \equiv 1 \pmod{p_2}.$$

Hence all factors of p are of the form $563x+1$.* There are no primes of this form less than the square root of p_2. Hence p_2 is a prime.

We find next that

$$2^{p_1-1} \equiv 4481813727 = 1 + 59306p_1 \pmod{p_1^2}$$

and

$$(2) \quad 2^{p_2-1} \equiv 70637882819917 = 1 + 3432372p_2 \pmod{p_2^2}.$$

Hence, by Theorem 3, equation (1) has no solutions x, y, z, prime to p for $p = p_1$ or p_2. We have then the following lemma.

THEOREM 4. *If $x^p+y^p+z^p=0$ has a solution for which xyz and p are coprime, then the first factor of the class number of the cyclotomic field $K(e^{2\pi i/p})$ is divisible by p^{12}.*

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* See this Bulletin, vol. 33 (1927), p. 331. Theorem 3.

ON CERTAIN CHARACTERISTICS OF k-DIMENSIONAL VARIETIES IN r-SPACE

BY B. C. WONG

An algebraic variety of k dimensions in r-space has numerous characteristics besides its order n. The characteristics of algebraic curves and surfaces and the relations they satisfy are known. In this paper we consider a variety V_k of dimension k greater than 2. Assuming it to be the complete intersection of $r-k$ hypersurfaces of orders $n_1, n_2, \cdots, n_{r-k}$ respectively in S_r, we derive the formulas for a few of its characteristics in terms of the n's and incidentally obtain the relations connecting them. To avoid unnecessary length of discussion we consider somewhat in detail the V_3 in S_7 only and then give the results without demonstration for V_k in S_r.* The method here employed is the familiar one of complete degeneration which we have repeatedly made use of elsewhere in dealing with problems of similar nature.†

Now for the purpose of enumerating the characteristics of V_k and obtaining their relations we may regard the variety as belonging to an S_{2k+1}, for a V_k belonging to an S_r where $r>2k+1$ possesses no characteristics not possessed by a V_k of S_{2k+1}. If we project V_k from a general S_{t-1} of S_{2k+1} on to an S_{2k+1-t} $[0\leqq t\leqq k]$ of S_{2k+1}, we have for projection a $V_k^{(t)}$ possessing a double $(t-1)$-dimensional variety $V_{t-1}^{b_{t-1}}$ of order b_{t-1} and a pinch $(t-2)$-dimensional variety $V_{t-2}^{j_{t-2}}$ of order j_{t-2} lying on $V_{t-1}^{b_{t-1}}$. From a general point of S_{2k+1-t} we can construct ∞^t lines forming a $(t+1)$-dimensional cone of order b_t each meeting $V_k^{(t)}$ in two distinct points. We say that $V_k^{(t)}$ has an apparent double $V_t^{b_t}$ of order b_t. Again, from a general point of S_{2k+1-t} a t-dimensional cone of ∞^{t-1} lines of order j_{t-1} can be constructed tangent to

* Some work has been done along this line. See C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, III$_2$, 7, pp. 922–927.

† B. C. Wong, *On the number of apparent multiple points of varieties in hyperspace*, this Bulletin, vol. 36, pp. 102–106; and *On surfaces in spaces of four and five dimensions*, this Bulletin, vol. 36, pp. 861–866.

$V_k^{(t)}$. The projection $V_k^{(t+1)}$ of $V_k^{(t)}$ in an S_{2k-t} has a double variety $V_t^{b_t}$ which is the projection of the apparent double variety $V_t^{b_t}$ of $V_k^{(t)}$ and a pinch variety $V_{t-1}^{j_{t-1}}$ which is the section of the tangent t-dimensional cone of $V_k^{(t)}$ by S_{2k-t}.

If we allow t to take on all the values from 0 to k, we obtain $2k$ characteristics $b_0, b_1, \cdots, b_{k-1}; j_0, j_1, \cdots, j_{k-1}$. For $t=0$, we have V_k itself, and it has b_0 apparent double points. For $t=1$, the projection V_k' in S_{2k} has b_0 improper double points and an apparent double curve of order b_1. From a general point of S_{2k} we can draw j_0 tangent lines to V_k'. Now if $t=2$, the projection V_k'' in S_{2k-1} has a double curve of order b_1 on which lie j_0 pinch points and an apparent double surface of order b_2, and is such that the cone of the ∞^1 tangent lines drawn from a general point of S_{2k-1} is of order j_1. And so on for the other values of t. For $t=k$, the projection $V_k^{(k)}$ in S_{k+1} has a double variety $V_{k-1}^{b_{k-1}}$ and a pinch variety $V_{k-2}^{j_{k-2}}$ lying on $V_{k-1}^{b_{k-1}}$. The characteristic j_{k-1} is the order of the tangent k-dimensional cone and it is also the class of the curve in which a plane of S_{k+1} meets $V_k^{(k)}$.

There are numerous other characteristics of the variety V_k such as the orders of its various manifolds of multplicities higher than 2 and the ranks of its different sections by subspaces of S_{2k+1}. With these we are not at present concerned and we are here concerned only with the b's and the j's just enumerated. Now we take the case $k=3$, that is, the V_3 in S_7.

By the method of complete degeneration we regard the four hypersurfaces of orders n_1, n_2, n_3, n_4 in S_7 which intersect in the V_3 we are studying as being composed entirely of hyperplanes: $A_1, A_2, \cdots, A_{n_1}$; $B_1, B_2, \cdots, B_{n_2}$; $C_1, \cdots, C_{n_3}$; $D_1, \cdots, D_{n_4}$. The V_3 of intersection is then composed of $n=n_1n_2n_3n_4$ 3-spaces $(A_{i_1}B_{i_2}C_{i_3}D_{i_4})$ $[i_j=1, 2, \cdots, n_j]$. We write $(x_1x_2x_3x_4)$ in place of $(A_{i_1}B_{i_2}C_{i_3}D_{i_4})$. This symbol for any set of particular values of the x's represents a particular S_3 belonging to the decomposed V_3. Thus, the set (1432) represents the S_3 common to A_1, B_4, C_3, D_2. The totality of all the $n=n_1n_2n_3n_4$ sets $(x_1x_2x_3x_4)$ for all the integral values of x_i from 1 to n_i $[i=1, 2, 3, 4]$ will be taken to be the symbolic representation of V_3.

To deal with the different apparent double varieties of V_3 is to deal with the different kinds of pairs of S_3's of the decomposed V_3, that is, with the different kinds of pairs of sets of values

$(x_1x_2x_3x_4)$. There are in all $n(n-1)/2$ pairs and they fall under four types. Under type I we have all those pairs each of which is such that the elements or x's of one set are all different from the corresponding elements or x's of the other, as for example (1111), (2324). Any such pair represents a pair of non-incident S_3's belonging to the decomposed V_3 and the number of such pairs is, as can easily be verified,

$$2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

Now type II consists of all those pairs in each of which three elements of one set are all different from the three corresponding elements of the other. Examples are (1111), (1223); (1234), (2414). A pair of this kind represents two S_3's in the decomposed V_3 having a point in common. The number of pairs belonging to this type is given by the symmetric function

$$2^2\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

For type III we have all those pairs each consisting of sets two of whose corresponding elements are different and two alike, as (1111), (1123). There are

$$2\sum n_1n_2\binom{n_3}{2}\binom{n_4}{2}$$

such pairs each representing two S_3's in the decomposed V_3 with a line in common.

The remaining

$$\sum n_1n_2n_3\binom{n_4}{2}$$

pairs of sets, that is, all those pairs each having three elements in one set and the three corresponding elements in the other alike, are said to belong to type IV. A pair of this type represents a pair of S_3's having a plane in common and belonging to the decomposed V_3.

Now from a given point in S_7 only one line can be drawn incident with two non-incident S_3's. We say that these two S_3's have a point of apparent intersection. The number of pairs of non-incident S_3's in the decomposed V_3 is the number of apparent double points of V_3 before decomposition. Being equal to the

number of pairs of sets of values of the x's belonging to type I, it is given by

$$b_0 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

Projecting the decomposed V_3 on to an S_5, we find that all those pairs of S_3's both non-incident and incident in a point project into pairs of S_3's each having an actual line of incidence. All the lines so obtained form the degenerate double curve in the projection of the decomposed V_3. Hence, the order of the double curve on the projection V_3'' of V_3 before degeneration is, from the results obtained for type I and type II,

$$b_1 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2} + 2^2\sum n_1\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}.$$

A pinch point on V_3'' is given by a tangent plane of V_3 passing through a given line l in S_7. If V_3 is completely degenerated, the given line l and the point of incidence of the two S_3's of a pair represented by a pair of sets of type II determine a plane which is to be regarded as two tangent planes of V_3 passing through l. Therefore, we have

$$j_0 = 2^3\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}$$

for the number of pinch points on the projection V_3'' in S_5.

Now a given plane of S_7 and the line common to two S_3's of a pair whose representation belongs to type III determine ∞^1 3-spaces each of which is to be considered as two tangent 3-spaces of V_3. If V_3 is projected on to an S_4, every such tangent 3-space meets S_4 in a pinch point of V_3'''. The order of the locus of pinch points on V_3''' is twice the number of pairs of sets of the x's belonging to type III and hence we have

$$j_1 = 2^2\sum n_1n_2\binom{n_3}{2}\binom{n_4}{2}.$$

In the projection in S_4 of the decomposed V_3 we find that the number of planes each of which is the projection of the plane of apparent intersection of the S_3's of a pair whose representation belongs to either type I or II or III is the order of the double surface on V_3''' and is given by

$$b_2 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2} + 2^2\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2} + 2\sum n_1n_2\binom{n_3}{2}\binom{n_4}{2}.$$

It is not difficult to see that the order j_2 of the tangent 3-dimensional cone of V_3''' is twice the number of pairs in type IV and we have

$$j_2 = 2\sum n_1n_2n_3\binom{n_4}{2}.$$

Incidentally we have the obvious relations

$$2b_1 = 2b_0 + j_0, \quad 2b_2 = 2b_1 + j_1 = 2b_0 + j_0 + j_1$$

and

$$n(n-1) = 2b_2 + j_2 = 2b_0 + j_1 + j_2.$$

It is to be noticed that if one of the four n's is unity, then $b_0=0$, that is, if V_3 is the complete intersection of three general V_5's in an S_6, it cannot have improper double points. If two of the four n's are equal to unity, then we have, in addition to b_0 vanishing, $b_1=0$ and $j_0=0$. This means that a V_3 of complete intersection of two general V_4's in an S_5 has no double curve nor pinch points.

Now we write the results for the b's and the j's of V_k in S_r for $r \geqq 2k+1$. If V_k is the complete intersection of $r-k$ general hypersurfaces of orders $n_1, n_2, \cdots, n_{r-k}$ respectively, we find by arguments exactly similar to those for the case $k=3$ the following formulas:

$$b_{k-q} = \sum_{i=1}^{j-k-q} 2^{q-1+i}\sum n_1n_2\cdots n_{r-k-q-i}\binom{n_{r-k-q-i+1}}{2}\cdot\binom{n_{r-k-q-i+2}}{2}\cdots\binom{n_{r-k}}{2},$$

and

$$j_{k-q} = 2^q\sum n_1n_2\cdots n_{r-k-q}\binom{n_{r-k-q+1}}{2}\binom{n_{r-k-q+2}}{2}\cdots\binom{n_{r-k}}{2}.$$

For $q=1$, we have

$$b_{k-1} = \sum_{i=1}^{r-k-1} 2^i \sum n_1 n_2 \cdots n_{r-k-1-i} \binom{n_{r-k-i}}{2} \binom{n_{r-k-i+1}}{2} \cdots \binom{n_{r-k}}{2}$$

$$= \tfrac{1}{2} n_1 n_2 \cdots n_{r-k} (n_1 n_2 \cdots n_{r-k} - \sum n_1 + r - k - 1),$$

which is the number of apparent double points on the curve in an S_2 into which the curve of intersection of V_k by an S_{r-k+1} is projected, and

$$j_{k-1} = 2 \sum n_1 n_2 \cdots n_{r-k-1} \binom{n_{r-k}}{2} = n_n \cdots n_{r-k} \sum (n_1 - 1),$$

which is the rank of the curve of intersection of V_k by an S_{r-k+1}.

By assigning to q all the different values from 1 to k, we obtain all the formulas for the $2k$ characteristics $b_0, b_1, \cdots, b_{k-1}$; $j_0, j_1, \cdots, j_{k-1}$. We have also the relations

$$\begin{aligned} 2b_{k-q} &= j_{k-q-1} + 2b_{k-q-1} \\ &= j_{k-q-1} + j_{k-q-2} + 2b_{k-q-2} \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ &\quad \cdots\cdots\cdots\cdots\cdots \\ &= j_{k-q-1} + j_{k-q-2} + \cdots + j_1 + j_0 + 2b_0, \end{aligned}$$

and

$$n(n-1) = j_{k-1} + j_{k-2} + \cdots + j_{k-q} + 2b_{k-q}.$$

We wish to add that, if $r < 2k+1$ and if $2k+1-r = d$, we have $b_0 = b_1 = \cdots = b_{d-1} = 0$ and also $j_0 = j_1 = \cdots = j_{d-2} = 0$.

THE UNIVERSITY OF CALIFORNIA

ON THE LEXIS THEORY AND THE ANALYSIS OF VARIANCE*

BY H. L. RIETZ

In a paper† published in 1921 involving a generalization of the Lexis theory for the classification of statistical series with regard to their dispersion into Bernoulli, Lexis, and Poisson series, Coolidge based much of his reasoning on the following fundamental dispersion theorem.

If n independent quantities $y_1, y_2, \cdots, y_n$ be given, their expected values being $a_1, a_2, \cdots, a_n$, while the expected values of their squares are $A_1, A_2, \cdots, A_n$, respectively, and if we agree to set $y = (1/n)\sum_{i=1}^{n} y_i$, $a = (1/n)\sum_{i=1}^{n} a_i$, then the expected value of the variance $(1/n)\sum_{i=1}^{n}(y_i - y)^2$ is

$$(1) \qquad \frac{1}{n}\left[\frac{n-1}{n}\sum_{i=1}^{n}(A_i - a_i^2) + \sum_{i=1}^{n}(a_i - a)^2\right].$$

In setting up criteria for the practical classification of actual statistical series, Coolidge followed the customary procedure of introducing approximations by replacing $(n-1)/n$ by 1.

By avoiding this approximation in the present paper but otherwise proceeding along the lines followed by Coolidge, we shall arrive at certain important results of R. A. Fisher in his analysis of variance. The different estimates of variance used by R. A. Fisher seem to have been obtained largely by inferences based on the number of degrees of freedom of the variates rather than upon formal mathematical proofs. In fact, the intuitional element is so prominent in certain of these inferences based on the number of degrees of freedom that mathematicians rather generally hesitate to accept the results as mathematically established, although there is much general evidence in favor of the correctness of the conclusions. For this reason, it seems of interest to show how certain of the results in question can be derived by formal developments that follow closely the reasoning of Coolidge in his generalization of the Lexis theory.

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* Presented to the Society, December 28, 1931.

† This Bulletin, vol. 27 (1921), p. 439.

The Lexis theory is concerned with the examination of the inner structure of a population of items by the separation of items into subsets. As a simple mode of subdivision, suppose the set of independent items classified in some relevant manner into N sets of s items each. Then our observations may well be exhibited in columns and rows as

$$
\begin{array}{l}
x_{11},\ x_{12},\ \cdots,\quad x_{1s},\ \bar{x}_{1\cdot} \\
x_{21},\ x_{22},\ \cdots,\quad x_{2s},\ \bar{x}_{2\cdot} \\
\quad\cdots\cdots\cdots\cdots \\
\quad\cdots\cdots\cdots\cdots \\
x_{N1},\ x_{N2},\ \cdots,\quad x_{Ns},\ \bar{x}_{N\cdot} \\
\bar{x}_{\cdot 1},\ \bar{x}_{\cdot 2},\ \cdots,\quad \bar{x}_{\cdot s},\ \bar{x}
\end{array} \tag{2}
$$

with arithmetic means $\bar{x}_{i\cdot}$ of the ith row, $\bar{x}_{\cdot j}$ of the jth column, and $\bar{x}$ of the whole sample of Ns items.

To illustrate, take $N=17$ and $s=15$. Suppose that $N=17$ years and that $s=15$ refers to the first fifteen days of each year and that x_{ij} gives the mimimal temperature in degrees Fahrenheit at a certain place for the jth day of the ith year.

Using $E(\ )$ for the expected value of the expression in the parenthesis, we let $E(x_{ij})=a_{ij}$, $E(x_{ij}^2)=A_{ij}$. Further, let

$$\sum_{j=1}^{s} a_{ij} = sa_i;\quad \sum_{i=1}^{N} a_i = Na;\quad \sum_{i=1}^{N} \bar{x}_{i\cdot} = N\bar{x};\quad \sum_{j=1}^{s} \bar{x}_{\cdot j} = s\bar{x}.$$

Then by (1)

$$E\sum_{j=1}^{s}(x_{ij}-\bar{x}_{i\cdot})^2 = \frac{(s-1)}{s}\sum_{j=1}^{s}(A_{ij}-a_{ij}^2) + \sum_{j=1}^{s}(a_{ij}-a_i)^2. \tag{3}$$

Summing (3) from $i=1$ to N, we have

$$E\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2 = \frac{(s-1)}{s}\sum_{i=1,j=1}^{N,s}(A_{ij}-a_{ij}^2) + \sum_{i=1,j=1}^{N,s}(a_{ij}-a_i)^2. \tag{4}$$

Next, we note that $E(\bar{x}_{i\cdot})=a_i$,

$$\bar{x}_{i\cdot} - a_i = \frac{1}{s} \sum_{j=1}^{s}(x_{ij} - a_{ij}), \qquad \text{and}$$

(5)

$$E(\bar{x}_{i\cdot} - a_i)^2 = \frac{1}{s^2} E\left\{ \sum_{j=1}^{s}(x_{ij} - a_{ij}) \right\}^2 = \frac{1}{s^2} \sum_{j=1}^{s}(A_{ij} - a_{ij}^2).*$$

Since $E(\bar{x}_{i\cdot}) = a_i$, we note that

$$E(\bar{x}_{i\cdot}^2) = E(\bar{x}_{i\cdot} - a_i)^2 + a_i^2. \tag{6}$$

By applying (1), we may write

$$E \sum_{i=1}^{N}(\bar{x}_{i\cdot} - \bar{x})^2 = \frac{N-1}{N} \sum_{i=1}^{N}[E(x_{i\cdot}^2) - a_i^2] + \sum_{i=1}^{N}(a_i - a)^2. \tag{7}$$

From (6) and (7), we have

$$E \sum_{i=1}^{N}(\bar{x}_{i\cdot} - \bar{x})^2 = \frac{N-1}{N} E \sum_{i=1}^{N}(\bar{x}_{i\cdot} - a_i)^2 + \sum_{i=1}^{N}(a_i - a)^2. \tag{8}$$

By substituting for $E(\bar{x}_{i\cdot} - a_i)^2$ in (8) from (5), we have

$$E \sum_{i=1}^{N}(\bar{x}_{i\cdot} - \bar{x})^2 = \frac{1}{s^2} \frac{N-1}{N} \sum_{i=1,j=1}^{N,s} (A_{ij} - a_{ij}^2) + \sum_{i=1}^{N}(a_i - a)^2. \tag{9}$$

Eliminate $\sum_{i=1,j=1}^{N,s} (A_{ij} - a_{ij}^2)$ from (4) and (9), and we obtain

$$E \sum_{i=1,j=1}^{N,s}(x_{ij} - \bar{x}_{i\cdot})^2 - \sum_{i=1,j=1}^{N,s} (a_{ij} - a_i)^2 = \frac{Ns(s-1)}{N-1} E \sum_{i=1}^{N}(\bar{x}_{i\cdot} - \bar{x})^2 - \frac{Ns(s-1)}{N-1} \sum_{i=1}^{N}(a_i - a)^2. \tag{10}$$

Bernoulli Series. In a Bernoulli series we assume statistical homogeneity in the sense that items are so thoroughly mixed that the expected value of any statistical estimate is independent of the portion of the population from which the sample is drawn. Measurements on the same quantity will serve as an illustration, the items differing only by accidental errors. Under these conditions, $a_{ij} = a_i$, $a_i = a$. Hence, we would compare

* See Coolidge, *Probability*, p. 63, Theorem 8.

$$(11)\qquad \sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2 \quad\text{and}\quad \frac{s(s-1)N}{N-1}\sum_{i=1}^{N}(\bar{x}_{i\cdot}-\bar{x})^2,$$

or

$$(12)\qquad \frac{1}{N(s-1)}\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2 \quad\text{and}\quad \frac{s}{N-1}\sum_{i=1}^{N}(\bar{x}_{i\cdot}-\bar{x})^2,$$

and expect to find them equal in a Bernoulli series except for sampling fluctuations. Those in (12) are frequently compared in the analysis of variance by the procedure of R. A. Fisher.

Lexis Series. The items within a set of s have the same expected value, but the expected values vary from one set to another. Thus, $a_{ij}=a_i$, but $a_i\neq a$. Then we expect

$$(13)\qquad \frac{1}{N(s-1)}\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2 < \frac{s}{N-1}\sum_{i=1}^{N}(\bar{x}_{i\cdot}-\bar{x})^2.$$

The series is said to have *supernormal* dispersion.

Poisson Series. There are differences in expected values within sets of s items, but all the N sets are comparable, that is, $a_{ij}\neq a_i$; but $a_i=a$. Then we expect

$$(14)\qquad \frac{1}{N(s-1)}\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2 > \frac{s}{N-1}\sum_{i=1}^{N}(\bar{x}_{i\cdot}-\bar{x})^2.$$

The series is said to have *subnormal* dispersion.

The expected values of the expressions in (12), (13), and (14) would be equal in a statistically homogeneous population to the expected value of $[1/(Ns-1)]\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x})^2$ which is ordinarily regarded as the best estimate of the population variance from a sample of Ns items of such a population; that is,

$$(15)\qquad \frac{1}{N(s-1)}E\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x}_{i\cdot})^2=\frac{1}{Ns-1}E\sum_{i=1,j=1}^{N,s}(x_{ij}-\bar{x})^2.$$

To establish (15), we may start with the fact that in a statistically homogeneous system of items

$$(16)\qquad \frac{1}{s-1}\sum_{j=1}^{s}(x_{ij}-\bar{x}_{i\cdot})^2$$

is our estimate of the population variance based on the use of s items drawn at random. Thus, (16) has the same expected value as the right member of (15). Furthermore, the arithmetic mean

$$(17) \qquad \frac{1}{N(s-1)} \sum_{i=1,j=1}^{N,s} (x_{ij} - \bar{x}_{i\cdot})^2$$

of N such independent values as (16) has the same expected value as (16). Hence (15) is established.

To summarize, we have shown with our subdivision of the items of the sample into rows and columns that the following estimates of variance have the same expected value, in a statistically homogeneous population:

$$(18) \qquad \begin{aligned} V &= \frac{S}{Ns-1}, \quad \text{where} \quad S = \sum_{i=1,j=1}^{N,s} (x_{ij} - \bar{x})^2, \\ V_i &= \frac{S_i}{N(s-1)}, \quad \text{where} \quad S_i = \sum_{i=1,j=1}^{N,s} (x_{ij} - \bar{x}_{i\cdot})^2, \\ V_j &= \frac{S_j}{s(N-1)}, \quad \text{where} \quad S_j = \sum_{i=1,j=1}^{N,s} (x_{\cdot j} - \bar{x}_{\cdot j})^2, \\ V_{i\bar{x}} &= \frac{sS_{i\bar{x}}}{N-1}, \quad \text{where} \quad S_{i\bar{x}} = \sum_{i=1}^{N} (\bar{x}_{i\cdot} - \bar{x})^2, \\ V_{j\bar{x}} &= \frac{NS_{j\bar{x}}}{s-1}, \quad \text{where} \quad S_{j\bar{x}} = \sum_{j=1}^{s} (\bar{x}_{\cdot j} - \bar{x})^2. \end{aligned}$$

We thus arrive at estimates of variance used by R. A. Fisher without making use of arguments involving the number of degrees of freedom of the items with which we are concerned. A comparison of the numerical values of $V_{i\bar{x}}$ and V_i by taking the ratio $V_{i\bar{x}}/V_i$ serves as an important step in solving the problem of testing for significant differences from row to row in (2).

Similarly, a comparison of the numerical values of $V_{j\bar{x}}$ and V_j by taking the ratio $V_{j\bar{x}}/V_j$ would be useful in testing the significance of differences from column to column in (2). In comparisons of two estimates of variance V' and V by means of the ratio V'/V, R. A. Fisher made a fundamental contribution to the theory of applied statistics by finding the distribution function of $z = \frac{1}{2} \log V'/V$ for the case of a normal parent distribution and showing it to be of such a nature that the significance of the discrepancy of z from expectation could be examined objectively and expressed in terms of odds in favor of or against a discrepancy as large as or larger than an assigned value.

THE UNIVERSITY OF IOWA

ON POLYNOMIALS IN A GALOIS FIELD*

BY LEONARD CARLITZ†

1. *Introduction.* Let p be an arbitrary prime, n an integer $\geqq 1$, $GF(p^n)$ the Galois field of order p^n; let $\mathfrak{D}(x, p^n)$ denote the totality of *primary* polynomials in the indeterminate x, with coefficients in $GF(p^n)$, that is, of polynomials such that the coefficient of the highest power of x is unity. In this note we give a number of miscellaneous results concerning the elements of $\mathfrak{D}$. The results are of two kinds. The first involve generalizations of certain formulas treated by the writer in another paper.‡ Thus if we let $\tau^{(\alpha)}(E)$ denote the number of divisors of E of degree α, then, for $\alpha \leqq \beta$ and $\alpha+\beta \leqq \nu$, ν the degree of E (we may evidently assume without any loss in generality that $\alpha, \beta \leqq \nu/2$),

$$\sum \tau^{(\alpha)}(E)\tau^{(\beta)}(E) = (\alpha+1)p^{n\nu} - \alpha p^{n(\nu-1)}, \tag{1}$$

the summation on the left being taken over all polynomials E of degree ν. The other results of this kind involve generalized totient functions, as defined in §4.

The second group of formulas are of a different nature. Let us write p_0 for p^n, and define

$$F_\rho(\nu) = \prod_{\alpha=1}^{\nu}(x^{p_0^\alpha} - x)^{p_0^{\rho(\nu-\alpha)}}, F(\nu) = F_1(\nu).$$

Then we show that the least common multiple of the polynomials of degree ν is

$$L(\nu) = F_0(\nu); \tag{2}$$

the product of all the polynomials of degree ν is

$$\prod_{\deg E=\nu} E = F(\nu) = F_1(\nu); \tag{3}$$

if $Q_\rho(\nu)$ denote the product of those polynomials of degree ν that

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† International Research Fellow.

‡ *The arithmetic of polynomials in a Galois field*, American Journal of Mathematics, vol. 54 (1932), pp. 39–50. Cited as A.P.

are not divisible by the ρth power of any polynomial (except 1), then

$$(4) \qquad Q_\rho(h\rho + k) = \frac{F(h\rho + k)}{F^{p_0}(h\rho - \rho + k)} \left\{ \frac{F_\rho^{p_0}(h-1)}{F_\rho(h)} \right\}^{\rho p_0^k},$$

where it is assumed that $0 \leqq k < \rho$.

2. *Notation.* Polynomials will be denoted by large italic letters, ordinary integers by small Greek and italic letters. We write deg E for the degree of the polynomial E;

$$|E| = p^{n\nu} = p_0^{\nu},$$

where $\nu = \deg E$. If s is a real quantity >1, then

$$\zeta(s) = \sum_E |E|^{-s},$$

summed over all E in $\mathfrak{D}$, is the zeta-function of $\mathfrak{D}$; and it is immediately verified that

$$(5) \qquad \zeta(s) = (1 - p_0^{1-s})^{-1}, \qquad p_0 = p^n.$$

3. *The τ-Functions.* We define

$$\sigma_t(E) = \sum_{A|E} |A|^t,$$

the summation being taken over all the divisors of E. Then we may verify without any difficulty the following $\mathfrak{D}$ analog of a well known Ramanujan identity:*

$$(6) \qquad \sum_E \frac{\sigma_t(E)\sigma_u(E)}{|E|^s} = \frac{\zeta(s)\zeta(s-t)\zeta(s-u)\zeta(s-t-u)}{\zeta(2s-t-u)}.$$

Now it is evident from the definition of $\tau^{(\alpha)}(E)$ and $\sigma_t(E)$ that

$$\sigma_t(E) = \sum_\alpha \tau^{(\alpha)}(E) p_0^{\alpha t}$$

so that the left member of (1) is the coefficient of $p_0^{\alpha t + \beta u - \nu s}$ in the right member of (6). But, using (5), the product of zetas in (6) is equal to

* Messenger of Mathematics, vol. 45 (1916), pp. 81–84, or Collected Papers, 1927, pp. 133–135, formula (15).

$$(7)\qquad \frac{1-p_0^{1+t+u-2s}}{(1-p_0^{1-s})(1-p_0^{1+t-s})(1-p_0^{1+u-s})(1-p_0^{1+t+u-s})}.$$

To determine the coefficient in question, we note first that, for $t, u<0, s>1$,

$$\frac{1}{(1-p_0^{1+t-s})(1-p_0^{1+u-s})(1-p_0^{1+t+u-s})}=\sum_{\alpha,\beta,\nu} p_0^{\alpha t+\beta u+\nu-\nu s},$$

where the sum on the right is extended over all $\alpha, \beta, \nu \geqq 0$, such that $\alpha, \beta \leqq \nu$, $\alpha+\beta \geqq \nu$. Then the denominator in (7) is

$$\sum_{\nu \geqq \alpha+\beta} \min\ (\alpha+1, \beta+1) p_0^{\alpha t+\beta u+\nu-\nu s}+\sum_{\nu<\alpha+\beta};$$

clearly the second sum contributes nothing to the coefficient of $p_0^{\alpha t+\beta u-\nu s}$ in (7) when $\nu \geqq \alpha+\beta$, and so may be ignored. The coefficient in question is therefore

$$\begin{cases}(\gamma+1)p_0^{\nu}-\gamma p_0^{\nu-1} & \text{for } \nu \geqq 2,\\ (\gamma+1)p_0^{\nu} & \text{for } \nu<2,\end{cases}$$

where $\gamma=\min(\alpha, \beta)$, thus completing the proof of (1).

By means of the Ramanujan identity (6) we may evidently evaluate

$$(8)\qquad \sum_{\deg E=\nu} \sigma_t(F)\sigma_u(F),$$

but for general t, u, the result is rather complicated. For certain special values of t, u, the sum in (8) is fairly simple. Thus, for $u=2t$, it may be verified that

$$(9)\qquad \sum_{\deg E=\nu} \sigma_t(E)\sigma_{2t}(E)=p_0^{\nu}\begin{bmatrix}\nu+3\\ 3\end{bmatrix}-p_0^{\nu-1+3t}\begin{bmatrix}\nu+1\\ 3\end{bmatrix},$$

where

$$\begin{bmatrix}\nu+3\\ 3\end{bmatrix}=\frac{(p_0^{(\nu+3)t}-1)(p_0^{(\nu+2)t}-1)(p_0^{(\nu+1)t}-1)}{(p_0^{3t}-1)(p_0^{2t}-1)(p_0^{t}-1)}.$$

Again, for $s=t=0$, if we put

$$\sigma_0(E)=\tau(E)=\sum_{A \mid E} 1=\sum_{\alpha} \tau^{(\alpha)}(E).$$

then it is obvious that (7) implies

$$\sum_{\deg E=\nu} \tau^2(E) = p^\nu \begin{bmatrix} \nu+3 \\ 3 \end{bmatrix} - p^{\nu-1} \begin{bmatrix} \nu+1 \\ 3 \end{bmatrix},$$

which is indeed a particular case of (9).

4. *Totient Functions.* Let $\phi(M; \alpha_1, \cdots, \alpha_k)$ denote the number of sets of (ordered) polynomials $A_1, \cdots, A_k$, such that

$$\deg A_i = \alpha_i, \qquad (A_1, \cdots, A_k, M) = 1.$$

Using this definition, we have evidently

$$\sum_{(A_1,\cdots,A_k,M)=1} |A_1|^{-s_1} \cdots |A_k|^{-s_k} = \sum_{\alpha_i=0}^{\infty} \phi(M; \alpha_1, \cdots, \alpha_k) p_0^{-\alpha_1 s_1 - \cdots - \alpha_k s_k}, \tag{10}$$

where the s_i are real and each >1. By means of this identity it is easy to express the general ϕ-function in simple terms. Let $f(s)$ denote the left member of (10); then since

$$\sum_{\text{all } A_i} |A_1|^{-s_1} \cdots |A_k|^{-s_k} = \prod_{P|M} \{1 + |P|^{-(s_1+\cdots+s_k)}\}^{-1} \sum_{(A_1,\cdots,A_k,M)=1} |A_1|^{-s_1} \cdots |A_k|^{-s_k}$$

it follows that

$$f(s) = \zeta(s_1) \cdots \zeta(s_k) \prod_{P|M} \{1 - |P|^{-(s_1+\cdots+s_k)}\},$$

where P runs through the irreducible divisors of M. Therefore, by (10) and (5),

$$\phi(M; \alpha_1, \cdots, \alpha_k) = p_0^{\alpha_1+\cdots+\alpha_k} \sum_{A|M}' \mu(A) |A|^{-k}, \tag{11}$$

the sum being taken over A, dividing M, and of degree $\leqq \min(\alpha_1, \cdots, \alpha_k)$. If all the quadratfrei divisors of M satisfy this condition, (11) may be written in the form

$$\phi(M; \alpha_1, \cdots, \alpha_k) = p_0^{\alpha_1+\cdots+\alpha_k} \prod_{P|M} (1 - |P|^{-k}). \tag{11'}$$

In particular, let $\alpha_1 = \cdots = \alpha_k = \nu$, the degree of M. We now write $\phi_k(M)$ in place of $\phi(M; \nu, \cdots, \nu)$, and (11) becomes

* $\mu(A)$ is the Möbius μ-function for $\mathfrak{D}$; see A.P., §4.

$$(12)\quad \phi_k(M) = |M|^k \prod_{P|M}(1 - |P|^{-k}) = \sum_{M=AB} \mu(A)|B|^k$$

(where now all the terms in both sum and product are included). It is clear either from the definition or from (12) that $\phi_k(M)$ is the $\mathfrak{D}$-analog of the Jordan ϕ-function of higher order.

5. *Sets of Relatively Prime Polynomials.* Let $\psi(\alpha_1, \cdots, \alpha_k)$ denote the number of sets of (ordered) polynomials $A_1, \cdots, A_k$, such that $\deg A_i = \alpha_i$, $(A_1, \cdots, A_k) = 1$. Then, clearly,

$$\sum_{\alpha_i=0}^{\infty} \psi(\alpha_1, \cdots, \alpha_k) p_0^{-(\alpha_1 s_1 + \cdots + \alpha_k s_k)} = \sum_{(A_1, \cdots, A_k)=1} |A_1|^{-s_1} \cdots |A_k|^{-s_k}$$

$$= \frac{\zeta(s_1) \cdots \zeta(s_k)}{\zeta(s_1 + \cdots + s_k)} = \frac{1 - p_0^{1-(s_1+\cdots+s_k)}}{(1 - p_0^{1-s_1}) \cdots (1 - p_0^{1-s_k})};$$

and therefore

$$(13)\quad \psi(\alpha_1, \cdots, \alpha_k) = \begin{cases} p_0^{\alpha_1+\cdots+\alpha_k}(1 - p_0^{1-k}) & \text{for } \alpha_1 \cdots \alpha_k \neq 0, \\ p_0^{\alpha_1+\cdots+\alpha_k} & \text{otherwise.} \end{cases}$$

As might be expected, the ϕ and ψ functions are closely related. Indeed, from the definition, $\psi(\alpha_1, \cdots, \alpha_k, \nu)$ is the number of sets of polynomials $A_1, \cdots, A_k, M$, such that

$$\deg A_i = \alpha_i, \deg M = \nu, (A_1, \cdots, A_k, M) = 1;$$

and therefore

$$(14)\qquad \psi(\alpha_1, \cdots, \alpha_k, \nu) = \sum_{\deg M=\nu} \phi(M; \alpha_1, \cdots, \alpha_k).$$

From (13) and (14) we have

$$(15)\quad \sum_{\deg M=\nu} \phi(M; \alpha_1, \cdots, \alpha_k) = \begin{cases} p_0^{\alpha_1+\cdots+\alpha_k+\nu}(1 - p_0^{-k}) \\ \qquad \text{for } \alpha_1 \cdots \alpha_k \nu \neq 0, \\ p_0^{\alpha_1+\cdots+\alpha_k+\nu} \text{ otherwise.} \end{cases}$$

In particular, if $\alpha_1 = \cdots = \alpha_k = \nu$, we get for the ϕ-function in (12)

$$\sum_{\deg M=\nu} \phi_k(M) = \begin{cases} p_0^{(k+1)\nu} - p_0^{k(\nu-1)+\nu} & \text{for } \nu > 0, \\ 1 & \text{for } \nu = 0. \end{cases}$$

6. *A Modification of the ϕ-Functions.* Let us now denote by

$\phi'(M; \alpha_1, \cdots, \alpha_k)$ the number of sets of *quadratfrei* polynomials $A_1, \cdots, A_k$, such that $\deg A_i = \alpha_i$, $(A_1, \cdots, A_k, M) = 1$. Then, as in §4, we show that

$$\sum \phi'(M; \alpha_1, \cdots, \alpha_k) p_0^{-(\alpha_1 s_1 + \cdots + \alpha_k s_k)} = \sum' |A_1|^{-s_1} \cdots |A_k|^{-s_k},$$

the sum on the right being taken over all *quadratfrei* A_i such that $(A_1, \cdots, A_k, M) = 1$; but this sum is equal to

$$\frac{\zeta(s_1) \cdots \zeta(s_k)}{\zeta(2s_1) \cdots \zeta(2s_k)} \prod_{P \mid M} (1 + |P|^{-(s_1 + \cdots + s_k)})^{-1}.$$

Therefore, if $\lambda(B)$ is the $\mathfrak{D}$-analog of the Liouville λ-function,* and if $q(\nu)$ is defined by the relation†

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{\nu=0}^{\infty} \frac{q(\nu)}{p_0^{\nu s}},$$

we have in place of (11)

$$(16) \qquad \phi'(M; \alpha_1, \cdots, \alpha_k) = \sum_B \lambda(B) q(\alpha_1 - \beta) \cdots q(\alpha_k - \beta),$$

the sum extending over all B whose irreducible divisors are divisors of M, and such that $\deg B = \beta \leqq \min(\alpha_1, \cdots, \alpha_k)$. As for the function of §5, let us define $\psi'(\alpha_1, \cdots, \alpha_k)$ to be the number of sets of *quadratfrei* polynomials $A_1, \cdots, A_k$, such that $\deg A_i = \alpha_i$, $(A_1, \cdots, A_k) = 1$. Then

$$\sum \frac{\psi'(\alpha_1, \cdots, \alpha_k)}{p_0^{\alpha_1 s_1 + \cdots + \alpha_k s_k}} = \frac{\zeta(s_1) \cdots \zeta(s_k)}{\zeta(2s_1) \cdots \zeta(2s_k)} \frac{\zeta(2s_1 + \cdots + 2s_k)}{\zeta(s_1 + \cdots + s_k)},$$

so that

$$(17) \qquad \psi'(\alpha_1, \cdots, \alpha_k) = \sum_{\beta} (-1)^{\beta} p_0^{\beta'} q(\alpha_1 - \beta) \cdots q(\alpha_k - \beta),$$

the sum being taken over all β, $0 \leqq \beta \leqq \min(\alpha_1, \cdots, \alpha_k)$; and β' is the greatest integer $\leqq (\beta+1)/2$.

Now, from the definition of ϕ' and ψ', it is clear that

$$(18) \qquad \psi'(\alpha_1, \cdots, \alpha_k, \nu) = \sum_{\deg M = \nu} \mu^2(M) \phi'(M; \alpha_1, \cdots, \alpha_k),$$

* That is, if $B = P_1^{e_1} P_2^{e_2} \cdots$, $\lambda(B) = (-1)^{e_1 + e_2 + \cdots}$; see A.P., §3.

† It is evident that $q(\nu) = p_0^{\nu} - p_0^{\nu-1}$ for $\nu \geqq 2$ and that $q(\nu) = p_0^{\nu}$ otherwise.

and therefore the sum

$$\sum_{\deg M=\nu}{}' \phi'(M; \alpha_1, \cdots, \alpha_k),$$

taken over quadratfrei M only, is equal to the right member of (17).

7. *The L.C.M. of Polynomials of Degree ν.* We recall the well known result that

$$x^{p^\nu} - x = \prod_{\alpha \mid \nu} \Theta(\alpha), \tag{19}$$

where $\Theta(\alpha)$ is the product of the irreducible polynomials of degree α. If now $L(\nu)$ is the L.C.M. of the polynomials of degree ν, it is evident, to begin with, that if P is irreducible of degree δ, then the exponent of the highest power of P dividing $L(\nu)$ is precisely $[\nu/\delta]$, the greatest integer $\leqq \nu/\delta$. Therefore

$$\begin{aligned} L(\nu) &= \prod_{\deg P \leqq \nu} P^{[\nu/\deg P]} \\ &= \prod_{\delta=1}^{\nu} \left\{ \prod_{\deg P=\delta} P \right\}^{[\nu/\delta]} = \prod_{\delta=1}^{\nu} \{\Theta(\delta)\}^{[\nu/\delta]}. \end{aligned} \tag{20}$$

On the other hand, by (19),

$$F_0(\nu) = \prod_{\alpha=1}^{\nu} (x^{p^\alpha} - x) = \prod_{\alpha=1}^{\nu} \prod_{\delta \mid \alpha} \Theta(\delta) = \prod_{\delta=1}^{\nu} \{\Theta(\delta)\}^{[\nu/\delta]}.$$

Comparison with the right member of (20) shows at once that

$$L(\nu) = F_0(\nu). \tag{2}$$

8. *The Product of Polynomials of Degree ν.* Formula (3) may be proved very quickly if we make use of the following theorem due to E. H. Moore:*

If G run through the linear forms $G=\alpha_0 x_0 + \cdots + \alpha_\nu x_\nu$, where the coefficients α_i lie in $GF(p^\nu)$, and the α_i of lowest subscript $\neq 0$ is taken $=1$, then

$$\prod G = \left| x_i^{p^{\nu j}} \right|, \qquad (i, j = 0, \cdots, \nu). \tag{21}$$

Suppose that in this theorem $x_i = x^{p^{\nu-i}}$ $(i=0, \cdots, \nu)$; then the left hand member of (21) has the value

* This Bulletin, vol. 2 (1896), p. 189.

$$\prod_{\alpha=0}^{\nu} \prod_{\deg E=\alpha} E = \prod_{\alpha=1}^{\nu} \prod_{\deg E=\alpha} E; \tag{22}$$

the right member of (21) is a familiar determinant, and is easily seen to be equal to

$$\prod_{\alpha=0}^{\nu-1} (x^{p_0^{\nu-\alpha}} - x)^{1+p_0+\cdots+p_0^{\alpha}}. \tag{23}$$

Therefore, comparing (22) and (23), we have at once the formula to be proved:

$$\prod_{\deg E=\nu} E = \prod_{\alpha=0}^{\nu-1} (x^{p_0^{\nu-\alpha}} - x)^{p_0^{\alpha}} = F(\nu). \tag{3}$$

9. *The Formula for $Q_\rho(\nu)$.* Since any E may be written in the form $E = GM^\rho$, $P^\rho \nmid G$, it is evident that, for $\nu = h\rho + k$, $0 \leqq k < \rho$,

$$\begin{aligned} F(\nu) &= \prod_{\deg E=\nu} E \\ &= \prod_{\alpha=0}^{h} \left\{ \prod_{\deg G=\nu-\alpha\rho} G^{p_0^{\alpha}} \right\} \left\{ \prod_{\deg M=\alpha} M^{\rho\, q_\rho(\nu-\alpha\rho)} \right\} \qquad (24) \\ &= \prod_{\alpha=0}^{h} Q_\rho^{p_0^{\alpha}}(\nu - \alpha\rho) \cdot \prod_{\alpha=0}^{h} \{F(\alpha)\}^{\rho\, q_\rho(\nu-\alpha\rho)}, \end{aligned}$$

where $q_\rho(\nu)$ is the number of polynomials E of degree ν such that $P^\rho \nmid E$ for any irreducible P. It is known that*

$$q_\rho(\nu) = \begin{cases} p_0^{\nu} - p_0^{\nu-\rho+1} & \text{for } \nu \geqq \rho, \\ p_0^{\nu} & \text{otherwise;} \end{cases}$$

so that

$$\sum_{\alpha=\beta}^{h} p_0^{\alpha-\beta} q_\rho(\nu - \alpha\rho) = p_0^{\nu-\beta\rho}. \tag{25}$$

Then the product in (24) is equal to

$$\begin{aligned} &\prod_{\alpha=1}^{h} \{F(\alpha)\}^{\rho\, q_\rho(\nu-\alpha\rho)} \\ &= \prod_{\alpha=1}^{h} \prod_{\beta=1}^{\alpha} (x^{p_0^{\beta}} - x)^{p^{\alpha-\beta} \cdot \rho\, q_\rho(\nu-\alpha\rho)} \end{aligned}$$

* A.P., §6.

$$= \prod_{\beta=1}^{h}(x^{p_0\beta} - x)^{\rho e_\beta}, \qquad e_\beta = \sum_{\alpha=\beta}^{h} p_0^{\alpha-\beta} q_\rho(\nu - \alpha\rho),$$

$$= \prod_{\beta=1}^{h}(x^{p_0\beta} - x)^{\rho p_0^{\nu-\beta\rho}} \quad \text{(by (25))}$$

$$(26) \qquad = \left\{ \prod_{\beta=1}^{h}(x^{p_0\beta} - x)^{p_0^{(h-\beta)\rho}} \right\}^{\rho p_0^k} = F_\rho^{\rho p_0^k}(h^n).$$

By (24) and (26)

$$F(\nu) = \prod_{\alpha=0}^{h} Q_\rho^{p_0^\alpha}(\nu - \alpha\rho) \cdot F_\rho^{\rho p_0^k}(h),$$

or, writing $h-\alpha$ for α,

$$(27) \qquad \prod_{\alpha=0}^{h} Q_\rho^{p_0^{h-\alpha}}(\alpha\rho + k) = F(h\rho + k) F_\rho^{-\rho p_0^k}(h)$$
$$= R_\rho(h\rho + k), \text{ say.}$$

It is now easy to evaluate Q_ρ. Indeed, substituting $h-1$ for h in (27), and raising both members of the resulting equation to the p_0th power, we have

$$\prod_{\alpha=0}^{h-1} Q_\rho^{p_0^{h-\alpha}}(\alpha\rho + k) = R_\rho^{p_0}(\nu - \rho),$$

and therefore

$$(4) \qquad Q_\rho(\nu) = R_\rho(\nu) R_\rho^{-p_0}(\nu - \rho).$$

It will be remarked that by (27) $R_\rho(\nu)$ is a polynomial, so that by (4), $Q_\rho(\nu)$ is expressed as the ratio of two polynomials.

From (27) we may deduce another result of some interest. Since no polynomial of degree $<\rho$ is divisible by the ρth power of an irreducible polynomial, it is evident that

$$Q_\rho(k) = F(k), \qquad (0 \leqq k < \rho):$$

therefore, by (27), the expression

$$F(h\rho + k) F^{-p_0^k}(k) F_\rho^{-\rho p_0^k}(h)$$

is a polynomial provided that $0 \leqq k < \rho$.

CAMBRIDGE, ENGLAND

ON EULER'S TOTIENT FUNCTION

BY D. H. LEHMER*

In this note we discuss the equation

$$k\phi(n) = n - 1, \tag{1}$$

where k is an integer, and $\phi(n)$ is Euler's totient function, giving the number of integers $<n$ and prime to n. Our main purpose is to show that if n is a solution of (1), then n is a prime or the product of seven or more distinct primes. One is tempted to believe the stronger statement that (1) has no composite solutions or, in other words, the integer n is a prime if (and only if) $\phi(n)$ divides $n-1$. We have not been able to establish this, however. The proof of the nonexistence of composite solutions of (1) seems about as remote as the proof of the nonexistence of odd perfect numbers and the two problems though not equivalent are not dissimilar.

Let n be a composite solution of (1) and let a be any number prime to n; then

$$a^{n-1} = (a^{\phi(n)})^k \equiv 1 \pmod{n},$$

so that n furnishes an example of the failure of the strict converse of Fermat's theorem for all values of a prime to n. This involves no contradiction, however. In fact $a^{560} \equiv 1 \pmod{561}$, for all a's prime to 561, although $561 = 3 \cdot 11 \cdot 17$.

Together with (1) we shall consider the equation

$$k\phi(n) = n + 1, \tag{2}$$

and show that it has exactly eight solutions if n has less than seven distinct prime factors. The case $k=1$ may be dispensed with since (2) has no solutions and (1) has a solution n, if and only if n is a prime. We first give a number of necessary conditions which any solution n of (1) or (2) must satisfy.

THEOREM 1. *If $n>2$, then n is a product of distinct odd primes.*

PROOF. From equations (1) and (2) it is obvious that n must be prime to $\phi(n)$, and since $\phi(n)$ is even for $n>2$, n must be odd.

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Also if n contained a square of a prime, $\phi(n)$ would be divisible by this prime. Hence n is a product of distinct odd primes.

We note in passing that $n=2$ is a solution of (1) or (2) with $k=1$, and $k=3$ respectively, and in what follows we shall therefore suppose that $n>2$ and hence odd.

THEOREM 2. *If p is a factor of n, then n contains no prime factor of the form $px+1$.*

PROOF. In fact if p and $px+1$ were prime factors of n, then n and $\phi(n)$ would have a factor p in common.

THEOREM 3. *If n is composite, it is a product of an even or odd number of prime factors of the form $4x-1$, according as it is a solution of* (1) *or* (2).

PROOF. This theorem follows from the fact that $\phi(n)$ and hence $n\pm 1$ must be a multiple of 4.

From Theorem 1 we can write

$$n = p_1p_2p_3 \cdots p_t,$$

where the p's are distinct odd primes, and we will suppose that

$$2 < p_1 < p_2 < \cdots < p_t.$$

We now consider solutions of (1) and (2) for different values of t.

CASE I. $t=1$. This case is easily disposed of. Since n is a prime, it is a solution of (1) only with $k=1$. Equation (2) becomes

$$k(n-1) = n+1 \text{ or } (k-1)(n-1) = 2.$$

Hence the only solutions are $n=2$, $k=3$ and $n=3$, $k=2$.

CASE II. $t=2$. This case is also quite simple. When $n=p_1p_2$, (1) becomes

$$k(p_1-1)(p_2-1) = p_1p_2-1 \text{ or } k-1 = \frac{1}{p_1-1}+\frac{1}{p_2-1}.$$

Hence $0<k-1\leqq\frac{1}{2}+\frac{1}{4}<1$, which is impossible since $k-1$ is an integer. The equation (2) for this case becomes

$$0 < k-1 = \frac{1}{p_1-1}+\frac{1}{p_2-1}+\frac{2}{(p_1-1)(p_2-1)},$$

the only solution being $p_1=3$, $p_2=5$, since $k-1<1$ when at least one prime exceeds 5.

Before proceeding to larger values of t we show that we can confine ourselves to the case $k=2$.

THEOREM 4. *If* $2<t\leqq 6$, *then* $k=2$.

PROOF. Solving (1) and (2) for k we have

$$k = \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \pm \frac{1}{\phi(n)}.$$

But since $t\geqq 3$, $n\geqq 105$, and $\phi(n)\geqq 48$. Also since $t\leqq 6$

$$k \leqq \frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{17}{16} + \frac{1}{48} = \frac{5715}{2048} < 3.$$

The somewhat wasteful inequalities used above may be considerably sharpened if necessary. The following will illustrate.

THEOREM 5. *If* n *is a solution of* (1) *or* (2) *for* $k=3$, *then* n *is a product of more than* 32 *distinct prime factors.*

PROOF. Since n is prime to $k=3$, the smallest prime factor of n is $\geqq 5$. By Theorem 4, $t>6$, so that $n \geqq 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$, so that $\phi(n)\geqq 18247680$. Then

$$3 = \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \pm \frac{1}{\phi(n)}$$

or

$$\prod_{i=1}^{t} \frac{p_i}{p_i - 1} \geqq 3 - \frac{1}{18247680} > 2.99;$$

but

$$\prod_{i=1}^{t} \frac{p_i}{p_i - 1} \leqq \frac{5\cdot 7\cdot 11\cdots q_t}{4\cdot 6\cdot 10\cdots (q_t - 1)} = \pi_t,$$

where q_t is the tth prime >3. Hence

$$\pi_t > 2.99.$$

Referring to Legendre's table* we see that $q_t>139$ or $t>32$, which proves the theorem.

THEOREM 6. *If* $2<t\leqq 6$, *then* n *is a multiple of* 15.

* *Théorie des Nombres*, vol. 1, 3d edition, Table IX; see also Glaisher, Messenger of Mathematics, vol. 28, p. 2.

PROOF. First suppose that n is prime to 3; then $n \geqq 5 \cdot 7 \cdot 11$ so that

$$k \leqq \frac{5}{4}\,\frac{7}{6}\,\frac{11}{10}\,\frac{13}{12}\,\frac{17}{16}\,\frac{19}{18} + \frac{1}{4 \cdot 6 \cdot 10} = \frac{324091}{184320} < 2,$$

which is absurd. Hence n is a multiple of 3. By Theorem 2, n contains no prime factor of the form $3x+1$. Suppose now that n is not divisible by 5; then $n \geqq 3 \cdot 11 \cdot 17$ and $\phi(n) \geqq 480$.

$$k \leqq \frac{3}{2}\,\frac{11}{10}\,\frac{17}{16}\,\frac{23}{22}\,\frac{29}{28}\,\frac{41}{40} + \frac{1}{480} = \frac{4184731}{2150400} < 2.$$

Hence the theorem. With this information we can dispose of the case $t=3$ using the following lemma.

LEMMA 1. *If n_0 is a solution of* (2) *and if p is any prime, then*

I. *$n = n_0 p$ is not a solution of* (1);

II. *$n = n_0 p$ is a solution of* (2) *if and only if $n_0 + 2 = p$.*

PROOF. In order that $n_0 p$ be a solution of (1) or (2)

$$k\phi(n) = k\phi(n_0 p) = k\phi(n_0)(p-1) = (n_0+1)(p-1) = n_0 p \pm 1.$$

Hence $p = n_0$ or $p = n_0 + 2$. The lemma follows since, by Theorem 1, $p \neq n_0$.

CASE III. $t=3$. By Theorem 6, $n = 15p$. But 15 is a solution of (2); hence, by Lemma 1, (1) has no solutions and (2) has the single solution $n = 3 \cdot 5 \cdot 17 = 255$. For $t = 4, 5, 6$, we prove the following lemma.

LEMMA 2. *If $k\phi(m) = m + \alpha$, and if p and q are primes for which*

$$k\phi(mpq) = mpq + \epsilon, \tag{3}$$

then

$$(\alpha p - m - \alpha)(\alpha q - m - \alpha) - (m^2 + \alpha m + \alpha\epsilon) = 0. \tag{4}$$

PROOF. Multiplying out the expression (4) and adding and subtracting αmpq we obtain

$$\alpha\{(m+\alpha)(p-1)(q-1) - mpq - \epsilon\} = \alpha\{k\phi(mpq) - mpq - \epsilon\}.$$

But from (3) this is zero, hence the lemma.

Solving (4) for $p-1$ we obtain

$$(4') \qquad p - 1 = \frac{mq + \epsilon}{\alpha q - k\phi(m)}.$$

CASE IV. $t=4$. From Theorem 6, $n=15pq$, $p<q$. Using Lemma 2 with $m=15$, $\alpha=1$, we get from (4)

$$(p - 16)(q - 16) = 240 + \epsilon.$$

For equation (1), $\epsilon=-1$. Since 239 is a prime, $p-16=1$, and $q-16=239$ or $q=255$, not a prime. For equation (2) $\epsilon=+1$. Since 241 is a prime, $p-16=1$ and $q-16=241$ or $p=17$ and $q=257$ gives a solution $n=3\cdot5\cdot17\cdot257$.

CASE V. $t=5$. Writing n in the form $n=3\cdot5\cdot pqr$, we first show that $p<53$. Making use of Theorem 2 and supposing that $p\geqq53$, we find

$$k \leqq \frac{3}{2}\,\frac{5}{4}\,\frac{53}{52}\,\frac{59}{58}\,\frac{83}{82} + \frac{1}{2\cdot4\cdot52\cdot58\cdot82} = \frac{973279}{494624} < 2.$$

Hence $p<53$. By Theorem 2, $p=17$, 23, 29, or 47. Writing $m=3\cdot5\cdot p$ we apply Lemma 2. Thus $\alpha=2\phi(m)-m=1$, 7, 13 or 31. For $\epsilon=-1$ equation (4) becomes, for these various cases,

$$(5) \qquad (q - 256)(r - 256) = 65279 = 29\cdot2251,$$

$$(6) \qquad (7q - 352)(7r - 352) = 121433 = 13\cdot9341,$$

$$(7) \qquad (13q - 448)(13r - 448) = 194867 = \text{prime},$$

$$(8) \qquad (31q - 736)(31r - 736) = 518849 = 211\cdot2459.$$

Since $q<r$, equation (5) implies $(q-256)=1$ or 29, and $r-256=65279$ or 2251. In the first case r is divisible by 5 and in the second case the same is true of q. Hence (5) has no solution (p, q) in primes. Taking (6), (7), and (8) modulo 7, 13, and 31, respectively, we see that these equations have no solutions, even in integers. For $\epsilon=+1$ the right sides of (5), (6), (7), and (8) become $65281=97\cdot673$, $121447=$ prime, $194893=79\cdot2467$, and $518911=$ prime. The first of these implies

$$\begin{cases} q - 256 = 1, \\ r - 256 = 65281, \end{cases} \quad \text{or} \quad \begin{cases} q - 256 = 97, \\ r - 256 = 673, \end{cases}$$

so that $q=257$, $r=65537$ or $q=353$, $r=929$ are solutions of (2). The other three equations are not solvable in integers. For Case V, then, there are no solutions of (1) and a pair of solutions of (2), namely

$$n = 3\cdot 5\cdot 17\cdot 257\cdot 65537 = 2^{32} - 1 = 4294967295,$$

$$n = 3\cdot 5\cdot 17\cdot 353\cdot 929 = 83623935.$$

CASE VI. $t=6$. This case involves so many trials that space can be given to only a brief account of the work. We find as before that $n=3\cdot 5\cdot pqrs$, where $p=17$, 23, 29, or 47. We shall first discuss the case $p=17$. In this case we can prove that $q\geqq 257$. In fact if $q<257$, then by Theorem 2, $q\leqq 239$ since 241 and 251 are of the form $5x+1$. Hence

$$k > \frac{3}{2}\,\frac{5}{4}\,\frac{17}{16}\left(\frac{239}{238}\right)^3 - \frac{1}{2\cdot 4\cdot 16(238)^3} = \frac{217577459}{107496151} > 2,$$

contrary to Theorem 4. Hence $q\geqq 257$. First let $q=257$ and apply Lemma 2 with $m=3\cdot 5\cdot 17\cdot 257$, $\alpha=1$. For $\epsilon=-1$, (4) becomes $(r-2^{16})(s-2^{16})=4294901759=19\cdot 181\cdot 1248881$. Hence $r=65555$, 65717, or 68975. But r is a prime. Therefore r can only be 65717. But this gives $s=23794275$ which is not a prime. Hence (1) has no solution of the form $n=3\cdot 5\cdot 17\cdot 257\cdot rs$. For $\epsilon=+1$, equation (4) becomes

$$(r - 2^{16})(s - 2^{16}) = 4294901761 = 193\cdot 22253377.$$

Hence $s=4294967297=2^{25}+1=641\cdot 6700417$, or

$$s = 22318913 = 3037\cdot 7349.$$

In neither case is s a prime. Hence (2) has no solution of the form $3\cdot 5\cdot 17\cdot 257\cdot rs$. We next let $q=263$ and obtain

$$(7r - 67072)(7s - 67072) = \begin{cases} 4498183673 = 2731\cdot 1647083 \\ \qquad \text{for } \epsilon = -1, \\ 4498183687 = 60337\cdot 74551 \\ \qquad \text{for } \epsilon = +1. \end{cases}$$

Again there is no solution. The same results are obtained for $q=293$, 317, 347. For $q=353$ we get

$$(97r - 90122)(97s - 90122) = \begin{cases} 4706165747 = \text{prime} \\ \qquad \text{for } \epsilon = -1, \\ 4706165773 = 6577\cdot 715549 \\ \qquad \text{for } \epsilon = +1. \end{cases}$$

For $\epsilon = -1$, $s = 83623935$ is not a prime. For $\epsilon = +1$ we find $r = 929$, $s = 83623937$, both being prime numbers. Hence (2) has the solution*

$$n = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 83623937 = 6992962672132095.$$

The same method shows that neither (1) nor (2) has a solution for $q = 359$ or 383.

It now becomes much easier to consider equation (4′) written in the form

$$s - 1 = \frac{255qr + \epsilon}{(q - 256)r - 256(q - 1)}, \tag{9}$$

where q is a fixed prime and r varies over primes between such limits as to make the denominator of (9) positive and still have $s > r$. By actual trial division the successive values of r are eliminated very rapidly. No further solutions of (1) and (2) of the form $3 \cdot 5 \cdot 17 \cdot qrs$ exist. The cases $p = 23$, 29 and 47 were dealt with by means of (4′) and no solutions of (1) and (2) were found.

Summing up, we have shown that (1) has no composite solutions involving fewer than 7 distinct prime factors while we have found the following solutions of (2):

$$2,\ 3,\ 3 \cdot 5,\ 3 \cdot 5 \cdot 17,\ 3 \cdot 5 \cdot 17 \cdot 257,\ 3 \cdot 5 \cdot 353 \cdot 929,$$
$$3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537,\ 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 83623937.$$

If (2) has any further solutions they are products of 7 or more prime factors.

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* This solution can be also obtained from the solution $n = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929$ by using Lemma 1. In the same way if 6992962672132097 is a prime

$$n = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 83623937 \cdot 6992962672132097$$
$$= 48901526933832864378258473353215$$

is a solution of (2).

ON THE RELATIONSHIP AMONG THE DIAGONAL FILES OF A PADÉ TABLE*

BY H. S. WALL

1. *Introduction.* The object of the following note is to investigate the relationship among the nth approximants of the different diagonal files of a Padé table; and to study the relationship among the limits of those files for a Stieltjes power series, in the case † that those files have different limits. We have found that an arbitrary file S_k converges to an expression of the form

$$\frac{\alpha_k p - \beta_k p_1}{\alpha_k q - \beta_k q_1}, \tag{1}$$

where p, p_1, q, q_1 are entire transcendental functions independent of k, and α_k, β_k are polynomials or constants. If we denote by u_k, v_k the numerator and denominator, respectively, of (1), then if k', k'' are two values of the index k, the following identity obtains:

$$u_{k'}v_{k''} - u_{k''}v_{k'} = \alpha_{k'}\beta_{k''} - \alpha_{k''}\beta_{k'}; \tag{2}$$

and the polynomial on the right is not identically zero if $k' \neq k''$.

2. *Preliminary Formulas.*‡ Let $\mathfrak{P}(x) = \sum_{v=0}^{\infty} c_v(-x)^v$ be a *normal* power series, and let $\mathfrak{E}(x) = \sum_{v=0}^{\infty} d_v(-x)^v$ be the reciprocal of $\mathfrak{P}(x)$. Set $\mathfrak{P}^{(k)}(x) = \sum_{v=0}^{\infty} c_{v+k}(-x)^v$, $\mathfrak{E}^{(k)}(x) = \sum_{v=0}^{\infty} d_{v+k}(-x)^v$, $k = 0, 1, 2, \cdots$. Then the series $\mathfrak{P}^{(k)}(x)$, $\mathfrak{E}^{(k)}(x)$ have corresponding continued fractions

$$\frac{1}{a_1^{(k)}} + \frac{x}{a_2^{(k)}} + \frac{x}{a_3^{(k)}} + \cdots, \quad \frac{1}{b_1^{(k)}} + \frac{x}{b_2^{(k)}} + \frac{x}{b_3^{(k)}} + \cdots,$$

respectively, where the numbers $a_n^{(k)}$, $b_n^{(k)}$ are different from 0.

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† Designated as "Case I" in the writer's paper, *On the Padé approximants associated with the continued fraction and series of Stieltjes*, Transactions of this Society, vol. 31 (1929), pp. 91–116. We show in the present article that *no two* of the diagonal files have the same limit, thus supplementing the earlier result.

‡ For details concerning the statements in this paragraph, see a paper by the writer in the Transactions of this Society, vol. 33 (1931), pp. 511–532.

The Padé approximants for $\mathfrak{P}(x)$ may be expressed in terms of the numerators and denominators of the nth convergents, $A_n^{(k)}/B_n^{(k)}$, $C_n^{(k)}/D_n^{(k)}$, of these continued fractions. Indeed if $[p, q] \equiv N_{p,q}/D_{p,q}$ is that Padé approximant in which the degrees of numerator and denominator do not exceed q and p, respectively, then we have the following formulas:

$$(3) \qquad [n-1, n+k-1] = \mathfrak{P}_k + (-x)^k A_{2n-1}^{(k)}/B_{2n-1}^{(k)},$$

$$(4) \qquad [n+k-1, n-1] = [\mathfrak{E}_k + (-x)^k C_{2n-1}^{(k)}/D_{2n-1}^{(k)}]^{-1},$$

$$(n-1, k = 0, 1, 2, \cdots),$$

where $\mathfrak{P}_k = \sum_{v=0}^{k-1} c_v(-x)^v$, $\mathfrak{E}_k = \sum_{v=0}^{k-1} d_v(-x)^v$, $\mathfrak{P}_0 = \mathfrak{E}_0 = 0$;

$$(5) \qquad [n, n-1] = A_{2n}/B_{2n}, \qquad [n-1, n] = D_{2n}/C_{2n},$$

$$(n = 1, 2, 3, \cdots).$$

In the right member of (3) there occur the polynomials $A_{2n-1}^{(k)}$, $B_{2n-1}^{(k)}$. In what follows we shall want to express these polynomials in terms of the polynomials A_m, B_m. For that purpose we have the following identities which we gave in the paper to which we referred at the beginning of this paragraph, namely

$$(6) \quad \begin{aligned} B_{2n-1}^{(k)} &= h_n^{(k-1)} B_{2n}^{(k-1)} - B_{2n+1}^{(k-1)}, \\ xA_{2n-1}^{(k)} &= c_{k-1}(h_n^{(k-1)} B_{2n}^{(k-1)} - B_{2n+1}^{(k-1)}) - (h_n^{(k-1)} A_{2n}^{(k-1)} - A_{2n+1}^{(k-1)}), \\ h_n^{(k-1)} B_{2n}^{(k)} &= B_{2n+1}^{(k-1)}, \\ xh_n^{(k-1)} A_{2n}^{(k)} &= c_{k-1} B_{2n+1}^{(k-1)} - A_{2n+1}^{(k-1)}, \end{aligned}$$

$(n, k = 1, 2, 3, \cdots)$. Here $h_n^{(k-1)} = a_1^{(k-1)} + a_3^{(k-1)} + \cdots + a_{2n+1}^{(k-1)}$, and is $\neq 0$. There are four similar relations for the $C_m^{(k)}$, $D_m^{(k)}$ of (4).

3. *The Diagonal Files S_k, $k > 0$.* We shall now turn to the Padé table, which is a table of double entry containing the approximant $[p, q]$ in the $(p+1)$th row and $(q+1)$th column. The fractions A_1/B_1, A_3/B_3, A_5/B_5, $\cdots$ constitute the principal diagonal file S_0, while S_{-1}, the first parallel file below S_0, is made up of the sequence A_2/B_2, A_4/B_4, A_6/B_6, $\cdots$. If k is any integer, then the diagonal file S_k is the sequence

$$S_k\colon [m, m+k], \quad \begin{cases} m = 0, 1, 2, \cdots, & \text{if } k \geqq 0, \\ m = -k, -k+1, -k+2, \cdots, & \text{if } k < 0. \end{cases}$$

The formulas (6) and (3) will enable us to express the approximants of a file S_k, $k>0$, in terms of the numerators and denominators of the fractions A_m/B_m of the files S_0 and S_{-1}. Thus if we set $k=1$ in the first two formulas of (6), we have, by (3),

$$[n-1, n] = \frac{h_n A_{2n} - A_{2n+1}}{h_n B_{2n} - B_{2n+1}}. \tag{7}$$

This is the nth approximant in the file S_1. Setting k successively equal to 1, 2 in (6), we may now express A''_{2n-1}, B''_{2n-1} in terms of the A_m, B_m. Then by (3) with $k=2$, we find that the nth approximant in the file S_2 is

$$[n-1, n+1] = \frac{[(h'_n/h_n) + x]A_{2n+1} - h_n A_{2n+2}}{[(h'_n/h_n) + x]B_{2n+1} - h_n B_{2n+2}}. \tag{8}$$

In the final step of the reduction we used the identities

$$A_{2n+1} = a_{2n+1}A_{2n} + xA_{2n-1},\quad B_{2n+1} = a_{2n+1}B_{2n} + xB_{2n-1}. \tag{9}$$

To obtain the general formula, suppose that, for a particular value of k,

$$[n-1, n+k-1] = \frac{M_k A_{2n+k-1} - N_k A_{2n+k}}{M_k B_{2n+k-1} - N_k B_{2n+k}}, \tag{10}$$

where M_k, N_k are polynomials in x in which the coefficients are rational functions of the quantities h_r^s. Let $\mathfrak{M}_k$, $\mathfrak{N}_k$ denote the polynomials obtained from these by replacing h_r^s by h_r^{s+1} throughout. Then*

$$[n-1, n+k] = c_0 - x\frac{\mathfrak{M}_k A'_{2n+k-1} - \mathfrak{N}_k A'_{2n+k}}{\mathfrak{M}_k B'_{2n+k-1} - \mathfrak{N}_k B'_{2n+k}}. \tag{11}$$

But when $k=2p$ this reduces, with the aid of (6), to (10) with $k=2p+1$, where

$$M_{2p+1} = \mathfrak{M}_{2p}h_{n+p}, \qquad N_{2p+1} = \mathfrak{M}_{2p} + (\mathfrak{N}_{2p}/h_{n+p}); \tag{12}$$

* This follows from the fact that the right member of (11) is a rational function of x in which the degrees of numerator and denominator do not exceed $n+k$ and $n-1$, respectively; and the expansion in ascending powers of x agrees with $\mathfrak{P}(x)$ for the first $2n+k-1$ terms. See Perron, *Die Lehre von den Kettenbrüchen*, Chapter X.

and when $k=2p-1$, (11) reduces to (10) with $k=2p$, where

$$(13)\quad M_{2p} = (\mathfrak{M}_{2p-1}/h_{n+p-1}) + xN_{2p-1}, \quad N_{2p} = N_{2p-1}h_{n+p-1}.$$

By (12) and (13) we see that M_{k+1} and N_{k+1} are polynomials of the same character as M_k and N_k, namely, they are polynomials in x in which the coefficients are rational functions of the quantities h_r^s. But this has been verified for small values of k, and is therefore universally true. Formulas (12) and (13), with the initial values obtainable from (7) and (8), may be used to compute successively the polynomials M_k and N_k.

4. *The Diagonal Files S_{-k}, $k>1$.* We shall next show that the approximants of the files S_{-k}, $k>1$, can be expressed in the form (10). We begin with the relations

$$\mathfrak{E}(x) \sim \cfrac{1}{b_1 + \cfrac{x}{b_2 + \cfrac{x}{b_3 + \cdots}}},$$

$$\mathfrak{P}(x) = \cfrac{1}{\mathfrak{E}(x)} \sim b_1 - \cfrac{x}{(-b_2) + \cfrac{x}{(-b_3) + \cfrac{x}{(-b_4) + \cdots}}},$$

from which it follows that

$$(14)\quad b_{2n} = -a'_{2n-1}, \quad b_{2n+1} = -a'_{2n}, \quad b_1 = c_0.$$

Also by symmetry we have

$$(15)\quad a_{2n} = -b'_{2n-1}, \quad a_{2n+1} = -b'_{2n}, \quad a_1 = d_0.$$

Furthermore

$$c_0 - xA'_n/B'_n = D_{n+1}/C_{n+1},$$

and consequently

$$c_0B'_n - xA'_n = \gamma_n D_{n+1},\ B'_n = \gamma_n C_{n+1},$$

where γ_n is independent of x. Giving to n the values $2p$ and $2p-1$ and employing (6) we then have

$$(16)\quad \begin{aligned} &A_{2p+1}/h_p = \gamma_{2p}D_{2p+1},\ B_{2p+1}/h_p = \gamma_{2p}C_{2p+1},\\ &h_pA_{2p} - A_{2p+1} = \gamma_{2p-1}D_{2p},\ h_pB_{2p} - B_{2p+1} = \gamma_{2p-1}C_{2p}. \end{aligned}$$

For the factors γ_n we have the following values:

$$(17)\quad \gamma_{2p} = 1,\ \gamma_{2p-1} = -1.$$

Indeed one may verify directly that the coefficients of the highest powers of x in A_{2p+1} and D_{2p+1} are 1 and $g_p = b_1 + b_3 + b_5 + \cdots + b_{2p+1}$, respectively. Hence by (14)

$$1/h_p = \gamma_{2p} g_p = \gamma_{2p}(c_0 - a_2' - a_4' - \cdots - a_{2p}').$$

But if we take $k=1$ in the first of the following known relations*

$$(18) \qquad a_{2n}^{(k)} = a_{2n+1}^{(k-1)}/h_{n-1}^{(k-1)} h_n^{(k-1)}, \quad a_{2n+1}^{(k)} = a_{2n+2}^{(k-1)}(h_n^{(k-1)})^2, \qquad (k = 1, 2, \cdots; a_n^0 = a_n),$$

this reduces to $1/h_p = \gamma_{2p}/h_p$, so that $\gamma_{2p} = 1$. In like manner we find that $\gamma_{2p-1} = -1$.

Let us now put $k=2$ in (4). Then as in (8) we find that, if we set $g_n^{(k)} = b_1^{(k)} + b_3^{(k)} + b_5^{(k)} + \cdots + b_{2n+1}^{(k)}$,

$$[n+1, n-1] = \frac{[(g_n'/g_n) + x]D_{2n+1} - g_n D_{2n+2}}{[(g_n'/g_n) + x]C_{2n+1} - g_n C_{2n+2}}.$$

But by (16) and (17), followed by an easy reduction in which the formulas (9), (14), and (15) play a part, this becomes

$$(19) \quad [n+1, n-1] = (g_n' A_{2n+1} + A_{2n+2})/(g_n' B_{2n+1} + B_{2n+2}),$$

which is the nth approximant in S_{-2}.

To obtain the general formula, we apply (10) to the series $\mathfrak{E}$, and use (4). This gives a relation of the form

$$[n+k-1, n-1] = \frac{V_k D_{2n+k-1} - W_k D_{2n+k}}{V_k C_{2n+k-1} - W_k C_{2n+k}},$$

where the V_k, W_k are polynomials in x with coefficients which are rational functions of the quantities g_r^s, and which may be calculated by means of formulas analogous to (12) and (13). By (16) and (17), we then have

$$(20) \qquad [n+k-1, n-1] = \frac{M_{-k} A_{2n+k-1} - N_{-k} A_{2n+k}}{M_{-k} B_{2n+k-1} - N_{-k} B_{2n+k}},$$

where

$$(21) \qquad \begin{aligned} M_{-2p} &= g_{n+p-1} V_{2p} - x W_{2p}, \quad N_{-2p} = -(W_{2p}/g_{n+p-1}), \\ M_{-2p-1} &= -V_{2p+1}/g_{n+p}, \quad N_{-2p-1} = g_{n+p} W_{2p+1} - V_{2p+1}. \end{aligned}$$

* Wall, Transactions of this Society, vol. 31 (1929), pp. 102–103.

The quantities M_{-k} and N_{-k} are polynomials in x with coefficients which are rational functions of the numbers g_r^s.

The recursion formulas analogous to (12) and (13) for the V_k and W_k may be combined with (21) to give the following recursion formulas for the polynomials M_{-k}, N_{-k}:

$$M_{-2p} = -g'_{n+p-1}\mathfrak{M}_{-2p+1}, \qquad N_{-2p} = \mathfrak{M}_{-2p+1} - \frac{\mathfrak{N}_{-2p+1}}{g'_{n+p-1}},$$

$$M_{-2p-1} = -\frac{\mathfrak{M}_{-2p}}{g'_{n+p-1}} + x\mathfrak{N}_{-2p}, \qquad N_{-2p-1} = -g'_{n+p-1}\mathfrak{N}_{-2p}. \tag{22}$$

Here the script letters have the same significance as before, namely $\mathfrak{M}_{-k}$, $\mathfrak{N}_{-k}$ are polynomials obtained by replacing g_r^s by g_r^{s+1} in the corresponding polynomials M_{-k}, N_{-k}.

5. *The Series of Stieltjes.* We shall now apply the work of the preceding paragraphs to the series of Stieltjes. If we set $x=1/z$ in $\mathfrak{P}(x)$ and in the corresponding continued fraction, and then divide by z, we will obtain the series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots, \tag{23}$$

and continued fraction

$$\cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}}. \tag{24}$$

These are the forms of the series and continued fraction found in Stieltjes' *Recherches sur les fractions continues.**

The numerators and denominators, $P_n(z)$ and $Q_n(z)$, of the nth convergents of (24) are connected with the $A_n(x)$, $B_n(x)$ by the following equations:

$$\begin{aligned} P_{2n}(z) &= z^{n-1}A_{2n}(1/z), \quad Q_{2n}(z) = z^n B_{2n}(1/z), \\ P_{2n+1}(z) &= z^n A_{2n+1}(1/z), \quad Q_{2n+1}(z) = z^{n+1}B_{2n+1}(1/z). \end{aligned} \tag{25}$$

When $a_n>0$, and $\sum a_n$ is convergent, Stieltjes found that

$$\begin{aligned} \lim_n P_{2n}(z) &= p(z), \qquad \lim_n Q_{2n}(z) = q(z), \\ \lim_n P_{2n+1}(z) &= p_1(z), \quad \lim_n Q_{2n+1}(z) = q_1(z), \end{aligned} \tag{26}$$

* Stieltjes, Oeuvres, vol. 2.

where $p(z)$, $p_1(z)$, $q(z)$, $q_1(z)$ are entire transcendental functions of genre 0, connected by the relation

$$p_1(z)q(z) - p(z)q_1(z) = +1. \tag{27}$$

Let us replace x by $1/z$ in (10) and (20), divide by z, and then introduce the polynomials $P_n(z)$ and $Q_n(z)$ of (25). After removing a common power of $1/z$ from numerators and denominators, these expressions then take the form

$$\frac{G_kP_{\delta-1} - H_kP_\delta}{G_kQ_{\delta-1} - H_kQ_\delta}, \tag{28}$$

where $\delta = 2n + |k|$, $(k = 0, \pm 1, \pm 2, \cdots)$. Here the G_k, H_k are polynomials in z given in terms of the M_k, N_k by the equations

$$G_k(z) = z^\theta M_k(1/z), \quad H_k = z^{\theta'} N_k(1/z), \qquad \theta' = \theta - [1-(-1)^k]/2, \tag{29}$$

where θ is the larger of the degrees of $M_k(x)$, $N_k(x)$ if k is even, and of $M_k(x)$, $xN_k(x)$ if k is odd. In particular, we have

$$G_0 = 1, H_0 = 0, G_{-1} = 1, H_{-1} = 0. \tag{30}$$

The others may be calculated successively by means of the recursion formulas (12), (13), (22). For example, we find that

$$\begin{aligned} &G_1 = h_nz, H_1 = 1; G_2 = (h_n'/h_n)z + 1, H_2 = h_nz, \\ &G_3 = (h_n''h_{n+1}/h_n')z^2 + h_{n+1}z, \\ &H_3 = [(h_n''/h_n') + (h_n'/h_{n+1})]z + 1, \\ &G_{-2} = g_n', H_{-2} = -1; G_{-3} = -(g_n''/g_n')z - 1, H_{-3} = g_n'. \end{aligned} \tag{31}$$

It is seen that in these cases

$$\left.\begin{aligned} G_{2p}(0) &= 1, & H_{2p}(0) &= 0, \\ G_{-2p-2}(0) &= g_{n+p}', & H_{-2p-2}(0) &= -1, \\ G_{2p+1}(0) &= 0, & H_{2p+1}(0) &= 1, \\ G_{-2p-3}(0) &= -1, & H_{-2p-3}(0) &= g_{n+p}'. \end{aligned}\right\} \quad (p = 0, 1, 2, \cdots). \tag{32}$$

The proof of (32) for all p is readily accomplished by mathematical induction.

Now when $a_n > 0$ and $\sum a_n$ converges, we see by (18) that $a_n' > 0$ and $\sum a_n'$ converges. Applying (18) again we find that

$a_n'' > 0$ and $\sum a_n''$ converges, etc. Also by (15) $b_n' < 0$ and $\sum b_n'$ converges. Then by (18) with $k=2$ and a_n', a_n'' replaced by b_n' and b_n'', respectively, it follows that $b_n'' < 0$ and $\sum b_n''$ converges, etc. Continuing in this way we find that all the series $\sum a_{2n+1}^{(k)}$, $\sum b_{2n+1}^{(k)}$ are convergent, and that their sums are >0 and <0, respectively. That is, there are finite numbers $h^{(k)} > 0$, $g^{(k+1)} < 0$ such that

$$\text{(33)} \quad \lim_n h_n^{(k)} = h^{(k)}, \ \lim_n g_n^{(k+1)} = g^{(k+1)}, \quad (k = 0, 1, 2, \cdots).$$

Turning now to the polynomials $G_k(z)$, $H_k(z)$ we find by (30) and (31) that for small values of k they converge, for $n=\infty$, to limiting forms which are polynomials of degree depending upon k, with positive coefficients if $k>0$, and with negative coefficients if $k<-1$. This same conclusion can be reached for all values of k by virtue of the recursion formulas (12), (13), and (22), in view of the relations (29). We shall express this result as follows:

$$\text{(34)} \quad \begin{aligned} &\lim_n G_{2p} = -\beta_{2p}, \quad \lim_n H_{2p} = -\alpha_{2p}, \\ &\lim_n G_{2p+1} = \alpha_{2p+1}, \quad \lim_n H_{2p+1} = \beta_{2p+1}, \\ &\qquad\qquad (p = 0, \pm 1, \pm 2, \pm 3, \cdots), \end{aligned}$$

where the $\alpha_k(z)$, $\beta_k(z)$ are polynomials in z. Furthermore by (32)

$$\text{(35)} \quad \begin{aligned} \beta_{2p}(0) &= -1, & \alpha_{2p}(0) &= 0; \\ \beta_{-2p-2}(0) &= -g', & \alpha_{-2p-2}(0) &= 1; \\ \beta_{2p+1}(0) &= 1, & \alpha_{2p+1}(0) &= 0; \\ \alpha_{-2p-3}(0) &= -1, & \beta_{-2p-3}(0) &= g', \end{aligned} \qquad (p \geqq 0).$$

Now by (34) and (26) it follows that the expression (28), which is the nth approximant in the kth diagonal file of the Padé table for the series (23), converges for $n=\infty$ to a limit u_k/v_k, where

$$u_k = \alpha_k p - \beta_k p_1, \ v_k = \alpha_k q - \beta_k q_1.$$

These functions satisfy the relation (2) as may be seen with the aid of (27). We state this result in the following theorem.

THEOREM 1. *Let* (23) *be a series of Stieltjes having a corresponding continued fraction* (24) *in which* $a_n > 0$. *Then if* $\sum a_n$ *converges, the kth diagonal file of the associated Padé table converges to the limit*

$$\frac{\alpha_k(z)p(z) - \beta_k(z)p_1(z)}{\alpha_k(z)q(z) - \beta_k(z)q_1(z)}, \tag{36}$$

where $p(z)$, $p_1(z)$, $q(z)$, $q_1(z)$ *are entire transcendental functions of* z, *and the* $\alpha_k(z)$, $\beta_k(z)$ *are polynomials in* z. *If* $u_k(z)$, $v_k(z)$ *denote the numerator and denominator, respectively, of the expression* (36), *then for two indices* k', k'' *we have the identity*

$$u_{k'}v_{k''} - u_{k''}v_{k'} = \alpha_{k'} - \alpha_{k''}\beta_{k'}. \tag{2}$$

The writer showed elsewhere that in the case under consideration, $u_{k'}/v_{k'} \not\equiv u_{k''}/v_{k''}$ if $k' \neq k''$ and $k', k'' \geqq 0$ or $k', k'' \leqq 0$. We are now able to prove the following supplementary theorem.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* $k' \neq k''$, *then* $u_{k'}/v_{k'} \not\equiv u_{k''}/v_{k''}$, *and therefore the right member of* (2) *is not identically* 0. *Furthermore, the functions* u_k/v_k *have a pole at* $z=0$ *if* $k \geqq 0$, *but are regular at* $z=0$ *if* $k<0$, *vanishing there if* $k<-1$.

In view of the earlier result mentioned above, it is clearly sufficient to prove here only the final statement in the theorem. But this follows at once from (35) and the equations*

$$p(0) = -g', \quad p_1(0) = 1, \quad q(0) = 1, \quad q_1(0) = 0.$$

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* See formulas for the $P_n(z)$, $Q_n(z)$ in Stieltjes' memoir, loc. cit. §2.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 29, 1932.

Abstracts must be in the hands of Associate Secretary Tomlinson Fort, 501 West 116th St., New York City, not later than October 8. In order to be printed in the BULLETIN in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than September 7. By invitation of the program committee, Professor Jesse Douglas will speak on *The problem of Plateau.*

AMES, IOWA, November 25-26, 1932.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 3. Abstracts received by that date will appear in the November issue of this BULLETIN. By invitation of the program committee, Professor E. B. Stouffer will speak on *Some canonical forms and their associated geometries,* and Professor H. W. March will speak on *Applications of the theory of elasticity to wood, a material of anistropic structure.*

ATLANTIC CITY AND PRINCETON, NEW JERSEY, ANNUAL MEETING, December 27-30, 1932.

Abstracts must be in the hands of the Secretary of the Society, 501 West 116th St., New York City, not later than November 30, 1932. Abstracts received by November 7 will appear in the November issue of the BULLETIN. There will be a joint session of Section A and The Mathematical Association, at which time Dean L. P. Eisenhart will deliver his retiring presidential address on the topic *Some recent developments in differential geometry,* and Professor E. R. Hedrick will give his retiring address as Chairman of Section A of the A.A.A.S., his topic being *Tendencies in the logic of mathematics.* There will be a joint session with the Econometric Society and Sections A and K. The tenth Josiah Willard Gibbs Lecture will be delivered by Professor R. C. Tolman of the California Institute of Technology on the topic *Thermodynamics and relativity.* On the afternoon of December 30, at Princeton University, there will be a symposium on *Applications of the operational calculus to mechanics,* at which time Professor J. von Neumann of Princeton University and the University of Berlin will be the main speaker and Professor G. D. Birkhoff will lead the discussion.

NEW YORK CITY, February 25, 1932.

Abstracts must be in the hands of the Secretary of the Society, 501 West 116th St., New York City, not later than February 4, 1932. Abstracts received by January 7 will appear in the January issue of the BULLETIN.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, October, 1930, p. 58), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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